THE THUAL-FAUVE PULSE: SKEW STABILIZATION

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ABSTRACT. Consider the quintic complex Ginzburg-Landau equation

$$u_t = (m + i\alpha\mu_0)u_{xx} - (m + i\alpha\mu_1)u + (1 + i\alpha\mu_2)|u|^2u - (1 + i\alpha\mu_3)|u|^4u, x \in \mathbb{R}.$$

The parameter m is close to $3/16$ so that for $\alpha = 0$, this equation possesses a unstable pulse-like solution. For $|\alpha|$ small the equation possesses pulse-like solutions of the form $e^{i\omega t}e^{i\phi(x)}r(x)$, with r a positive function decreasing exponentially at infinity and ϕ asymptotic to $-C|x| + D$ at infinity.

These solutions are linearly unstable for $|\alpha| \leq \alpha_c$; if a certain rational function of μ_2 and μ_3 is strictly positive and not too large, they become stable for $|\alpha| \geq \alpha_c$: when the initial data is a pulse plus a small perturbation, its limit for large times is the same pulse, possibly translated in space and in phase.

This article gives a rigorous proof of a conjecture of Thual and Fauve [41]; it relies on a very detailed asymptotic analysis of the eigenvalues of the linearized operator, depending on the parameters $3/16 - m$ and μ_j.

1. INTRODUCTION

In [41], Thual and Fauve proposed a model of localized structures generated by subcritical instabilities; in their article, they mentioned several examples of such localized structures in systems far from equilibrium: local regions of turbulent motion surrounded by laminar flow as in [42] chapter 19, spatially localized traveling waves at convection onset in binary fluid mixtures as in [32] or [17], a Faraday experiment in a narrow annular dish as in [29].

The equation proposed by Thual and Fauve is the quintic Ginzburg-Landau equation

$$\frac{\partial u}{\partial t} = m_0\frac{\partial^2 u}{\partial x^2} + m_1u + m_2|u|^2u + m_3|u|^4u, \tag{1.1}$$

where the m_i are complex coefficients with $\Re m_0 > 0$.

Thual and Fauve assumed $\Re m_3 < 0$, in order to stabilize large amplitudes, and $\Re m_1 < 0$, so that the zero solution should be stable. Choosing $\Re m_2 > 0$ and adequate relations on the other parameters of the equation ensured that there would exist non zero stable homogeneous solutions of the form $re^{i\omega t+kx}$.

They integrated numerically (1.1) and found thus a solution of (1.1) of the form

$$u(x,t) = e^{i\omega t}r(x)e^{i\phi(x)}, \tag{1.2}$$

where r takes positive values and decays exponentially at infinity, while the phase ϕ is asymptotic to $-C|x| + D$ at $|x| = \infty$. They obtained numerically standing wave solutions of (1.3), which they found to be experimentally stable for large enough absolute values of the imaginary parts of m_1, m_2 and m_3.

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Let us first simplify (1.1) thanks to some scale consideration: by changing the units of t, u and x, we can see that it is possible to choose $\Re m_2 = 1$, $\Re m_3 = -1$ and $m = \Re m_0 = -\Re m_1$. Thanks to the S^1 equivariance, we can also take m_0 to be real, with very little loss of generality. For reasons which will be justified in the course of the article, our results hold when m is slightly less than $3/16$; let it be said only at this point that for $m = 3/16$, and all the m_i's real, there exist four real heteroclinic solutions of

$$-mr'' + mr - r^3 + r^5 = 0,$$

which are distinct up to translation. We let $\tilde{r}$ be one of these heteroclinic solutions which takes the value $\sqrt{3}/2$ at $-\infty$ and 0 at $+\infty$; the three other ones are obtained by mirroring $\tilde{r}$, x or both.

For m slightly less than $3/16$, there are two homoclinic solutions with a very large "shelf", i.e. a region where the solution is very close to $\pm 3/4$; it is the presence of this large shelf which makes this choice of parameters interesting.

For real values of m_i, pulses must be linearly unstable. Let us sketch the argument which proves this statement: if u is a pulse-like solution according to the above definition, it will be proved at lemma 2.1 that u can be taken real, even and positive; then u solves the following differential equation:

$$-mu_{xx} + mu - u^3 + u^5 = 0, \quad x \in \mathbb{R}. \tag{1.3}$$

Moreover, the linearized operator at u is the unbounded operator in $L^2(\mathbb{R})$ given by

$$D(A) = H^2(\mathbb{R}), \quad As = -ms'' + ms - 3u^2 s + 5u^4 s. \tag{1.4}$$

The operator A can also be seen as a Schrödinger operator in $\mathbb{R}$, with potential $m - 3u^2 + 5u^4$; in particular, its essential spectrum lies above m. Differentiating (1.3) with respect to x, we find

$$Au' = 0,$$

which expresses the translation invariance of (1.3). Thus u' is an eigenfunction relative to the eigenvalue 0; since u' changes sign, the maximum principle implies that 0 cannot be the lower bound of the spectrum of A. We denote by $\lambda < 0$ this lower bound.

The eigenmode pertaining to λ is the shrinking-swelling mode: under most perturbations, the pulse shrinks to 0 or swells to infinity.

In order to prove our main results, we introduce a few notations:

$$\nu = 1 - \frac{16m}{3}, \tag{1.5}$$

$$L = \frac{1}{4} \ln \frac{4}{\nu}. \tag{1.6}$$

In the first part of this introduction, we restrict ourselves to a simplified case of (1.1):

$$u_t = -mu + mu_{xx} + (1 + i\alpha)|u|^2 u - |u|^4 u, \tag{1.7}$$

where m and α are real parameters;

In this article, we prove the following existence theorem:

Theorem (Theorem 4.5)**.** *For all $p > 0$, for all ν small enough, there exists $C > 0$ such that for all α satisfying*

$$|\alpha| \le \alpha_m = \frac{1}{2}\sqrt{\nu}\left(1 - CL^{-p}\right),$$

there exists a pulse solution of (1.7)*, i.e. a solution of the form*

$$u(x,t) = e^{i\omega t} r(x) e^{i\phi(x)},$$

where r is a positive function which decays exponentially at infinity, and $\phi(x)$ is asymptotic at infinity to $-C|x| + D$, with C a positive constant and D a real constant.

Then, we prove the stabilization property conjectured by Thual and Fauve:

Proposition (Proposition 5.8)**.** *Let u be the solution defined at theorem 4.5. Let $|\alpha| \le \alpha_m$ be the above defined number; there exists a number α_c whose asymptotic is*

$$\alpha_c(\nu) = \frac{1}{2}\sqrt{\nu}\left(1 - \frac{\pi^2}{48L^2} + O\big(L^{-5/2}\big)\right).$$

such that the solution u is stable iff $\alpha_c < |\alpha| < \alpha_m$, and unstable if $|\alpha| \le \alpha_c$.

More precisely, if $|\alpha| < \alpha_c$, the spectrum of the linearized operator at u contains exactly one negative eigenvalue; when $\alpha = \alpha_c$, the eigenvalue 0 is of algebraic multiplicity 3 and geometric multiplicity 2, with a non trivial Jordan block of dimension 2; when $\alpha_c < |\alpha| < \alpha_m$, the linearized operator at u has the semisimple double eigenvalue 0, and the remainder of the spectrum is bounded away from the imaginary axis.

The general case i.e.

$$u_t = m u_{xx} - (m + i\alpha\mu_1)u + (1 + i\alpha\mu_2)\,|u|^2\, u - (1 + i\alpha\mu_3)\,|u|^4\, u. \tag{1.8}$$

is treated in section 6 with very few analytical details; under the condition

$$\chi(\alpha) = \left[\mu_2 - \frac{9\mu_3}{8}\right]\left[\frac{\pi^2\mu_2}{4} - \frac{3\pi^2\mu_3}{16} + \frac{9\mu_3}{16}\right]\frac{1}{\big(2\mu_2 - 15\mu_3/8\big)^2} > 0,$$

skew stabilization also takes place, as is shown in Proposition 6.1.

Let us give a very rough idea of the reason for existence, and of the skew stabilization mechanism. In both cases, we will restrict ourselves to the simplified equation (1.7).

For the existence, we start from so-called kinks: they are solutions of (1.7) of the form

$$u(x,t) = e^{i\omega t} K(x - ct), \tag{1.9}$$

with

$$K(x) = r(x) e^{ikx}. \tag{1.10}$$

to make things precise, we demand that r take positive values, increasing from $r(-\infty) = 0$ to $\bar r = r(+\infty) > 0$. If we substitute (1.9) and (1.10) into (1.7), we obtain the equation

$$i\omega r - c(r' + ikr) - m(r'' + 2ikr' - k^2 r) + mr - (1 + i\alpha)r^3 + r^5 = 0. \tag{1.11}$$

Define $\tilde\omega = \omega - kc$; the vanishing of the imaginary part of (1.11) implies that

$$r' = \frac{(\tilde\omega - \alpha r^2) r}{2k}. \tag{1.12}$$

Since we assumed that r is increasing from 0 to $\bar r$ and is strictly positive, relation (1.12) implies the sign conditions:

$$\tilde\omega / k > 0, \quad \alpha / k > 0. \tag{1.13}$$

If we differentiate this relation with respect to x, we obtain

$$r'' = \frac{(\tilde\omega - \alpha r^2)(\tilde\omega - 3\alpha r^2) r}{4k^2}. \tag{1.14}$$

Substituting the expressions (1.12) and (1.14) in the real part of (1.11), we obtain a polynomial of degree 5 in r; since r is assumed to be different from 0, we obtain through straightforward algebra

$$k^2 = \frac{3\alpha^2}{4m}, \quad 2\tilde{\omega} + ck = \frac{3\alpha}{2}, \quad \tilde{\omega}^2 - \alpha\tilde{\omega} + \alpha^2(m + 3\alpha^2/4) = 0.$$

The second degree equation in $\tilde{\omega}$ has real roots if and only if $4\alpha^2 - \nu < 1/3$, which we assume from now on. These roots are given by

$$\bar{\omega} = \frac{\alpha}{4}\left(2 \pm \sqrt{1 + 3(\nu - 4\alpha^2)}\right). \tag{1.15}$$

We see also that

$$\bar{r} = \sqrt{\tilde{\omega}/\alpha}. \tag{1.16}$$

If we linearize the equation for $v = u(x - ct)\exp\bigl(-ik(x - ct) - i\omega t\bigr)$ around $\bar{r}$, the condition for stability of 0-wave number modes is

$$2\bar{r}^2 - 4\bar{r}^4 < 0.$$

Comparing this relation with (1.16), we can see that we have to choose the + sign in (1.15).

Hence, the velocity c is given by

$$c = \frac{\sqrt{3}(4\alpha^2 - \nu)}{1 + \sqrt{1 + 3(\nu - 4\alpha^2)}}.$$

where we have used the sign condition (1.13).

If $4\alpha^2 > \nu$, the velocity c is positive, and the zero state gains over the non zero state; we will say that this is the inflow situation; on the contrary, if $4\alpha^2 < \nu$, the non zero state gains over the zero state; we will say that we have an outflow situation.

Assume from now on that α and ν are small. If there existed a pulse-like solution, it could probably be approximated by a combination of a kink $K(x + L - ct)e^{i\omega t}$ centered at $-L$ and an antikink $K(L - x - ct)$ centered at L, provided that we know how to glue their phases together. In the outflow case, two competing effects take place: on one hand, the choice of parameters tends to produce an expanding pulse; on the other hand, an attraction effect between the walls is expected as in [5], [11] and [10]. It is natural to expect that this attraction effect should be exponentially small with L. Thus, it is reasonable to conjecture that the evolution of the pulse will be given by

$$\dot{L} = -c - C_1 e^{-C_2 L}.$$

The pulse will be in an equilibrium if the two competing effects balance, i.e.

$$L \sim \frac{1}{C_2}\ln\frac{1}{\nu - 4\alpha^2}. \tag{1.17}$$

This analysis assumes an almost scalar pulse, so that it is easy to glue together the phase of the kink and of the antikink. Of course, the pulse obtained by this argument is not stable: any amount of swelling or shrinking of L destabilizes it.

Let us consider the inflow case: $4\alpha^2 > \nu$. Now, nothing scalar can stop the pulse from collapsing. However, Malomed and Nepomnyashchy argue in [30] that the pulse does not collapse because of phase incompatibility: a minimum distance is necessary to match the phases of the kink and of the antikink. By formal asymptotics, they obtain a half length of the pulse given by

$$L \sim \frac{C_3}{4\alpha^2 - \nu}, \tag{1.18}$$

in a very small range $\alpha^4 \ll 4\alpha^2 - \nu \ll \alpha^2$.

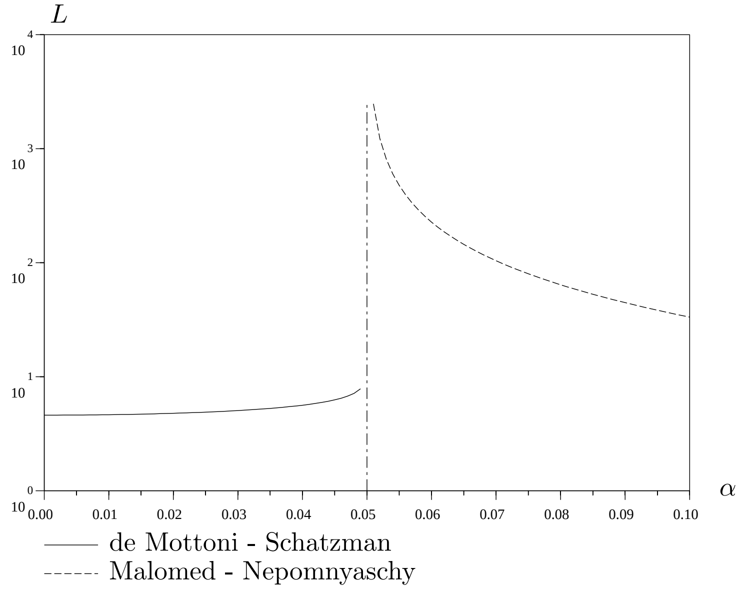

FIGURE 1. The length L of the pulse as a function of α..

However, the second author of the present article tends to believe that (1.18) holds for a much larger range of α. The article [30] contains no analysis of the stability of the pulse obtained; however the claim of stability (in a mathematical sense) seems quite reasonable.

With our results and the results of Malomed and Nepomnyashchy, we can plot a graph of L as a function of α, and we obtain Fig. 1.

The region around $4\alpha^2 = \nu$ seems difficult and interesting; jumping somewhat too fast to conclusions, it would be nice to believe that these two branches join to form a single branch; proving this might mean considerable effort.

Let us now give an idea of the mechanism of skew stabilization. We recall that we assumed that the shelf in the solution is very large, i.e. ν is very small. The definition of L is such that for $\alpha = 0$, the width of the shelf of the pulse is approximately $2L$. We define a function

$$F(u, \omega, \alpha, \nu) = i\omega u - mu'' + mu - (1 + i\alpha)|u|^2 u + |u|^4 u.$$

A function $ue^{i\omega t}$ is a solution of (1.7) iff u and ω are such that $F(u, \omega, \alpha, \nu)$ vanishes. Moreover, it is linearly stable if $D_u F(u, \omega, \alpha, \nu)$ has its spectrum included in the right-hand side complex half-plane; moreover, we require for stability that 0 should be a semi-simple eigenvalue of finite multiplicity while the remainder of the spectrum is bounded away from the imaginary axis.

The symmetries of the problem imply that there are two eigenfunctions of $D_u F$, relative to the eigenvalue 0: $t = iu$ and u'; there is also the eigenfunction w relative to the shrinking-swelling mode, and the aim of the game is to show that the corresponding eigenvalue λ crosses 0 for some appropriate value of α. Let us define

$$\tilde{F}(u, \tau, L, \beta) = F\big(u, e^{-2L}\beta\tau, e^{-2L}\beta, e^{-4L}\big).$$

At this point, we will use a measure of cheating in order to explain what is going on: if u and τ can be seen as smooth functions of L, we differentiate with respect

to L, and we find

$$\text{(1.19)}\quad \begin{cases} D_u\tilde{F}(u(\cdot,L,\beta),\tau(L,\beta),L,\beta) = e^{-2L}\beta\dfrac{\partial\tau}{\partial L}u + 2ie^{-2L}\beta\bigl(\tau u - |u|^2u\bigr) + \\ \qquad\qquad\qquad \dfrac{4\nu}{1-\nu}\bigl[ie^{-2L}\beta\tau u - |u|^2u + |u|^4u - ie^{-2L}\beta|u|^2u\bigr]. \end{cases}$$

Since we expect the interaction between the rotation mode and the shrinking-swelling mode to be the reason for the skew stabilization, we let $\{\hat{w},\hat{t}\}$ be a dual basis to the basis $\{w,t\}$, i.e.

$$\begin{aligned} &\int \hat{t}^T t\,dx = \int \hat{w}^T w\,dx = 1, \quad \int \hat{t}^T w\,dx = \int \hat{w}^T t\,dx = 0,\\ &y \text{ and } t \text{ are even, } \hat{t}^T D_u\hat{F}(u(L,\beta),\tau(L,\beta),L,\beta) = 0,\\ &\hat{w}^T D_u F(u(L,\beta),\tau(L,\beta),L,\beta) = \hat{w}^T\lambda(L,\beta). \end{aligned}$$

We multiply (1.19) on the left by $\hat{w}^T$, we integrate, and we find that

$$\begin{aligned} \lambda(L,\beta)\int \hat{w}^T\frac{\partial u}{\partial L}\,dx &= -2e^{-2L}\beta\int \hat{w}^T i|u|^2u\,dx\\ &\quad + \frac{4\nu}{1-\nu}\int \hat{w}^T\bigl[-|u|^2u + |u|^4u - ie^{-2L}\beta|u|^2u\bigr]\,dx. \end{aligned}$$

Since α is small, the problem is very close to being self-adjoint, it is not unreasonable to take w as an approximation of $\hat{w}$; moreover, in the approximation of the large shelf, u is very close to $\tilde{r}(|x|-L)$; therefore, w and $\partial u/\partial L$ can be taken very close to $-\tilde{r}'(|x|-L)\operatorname{sgn} x$. These considerations imply that

$$\int \hat{w}^T\frac{\partial u}{\partial L}\,dx \sim 2\int_{-L}^{\infty}|\tilde{r}'|^2\,dx = \frac{3}{8}.$$

We see also that

$$\int \hat{w}^T\bigl[-|u|^2u + |u|^4u\bigr]\,dx \sim 2\int_{-L}^{\infty}\tilde{r}'\tilde{r}^3 - \tilde{r}'\tilde{r}^5\,dx \sim -\frac{9}{64}.$$

Thus, we obtain the following "equivalent" for $\lambda(L,\beta)$:

$$\lambda(L,\beta) \sim -\frac{3e^{-4L}}{2} - 2e^{-2L}\beta\int \hat{s}^T i|u|^2u\,dx.$$

If we are able to calculate with enough precision the integral in the right hand side of the above equation, and if it turns out to be negative, we will hope for a stabilization effect.

However, we have cheated too much for this argument to go through a rigorous analysis; the main source of inexactitude comes from the assumption that for α small, we can find an even solution u of $\tilde{F}(u,\tau,L,\beta)$ which is close to the real even solution of $\tilde{F}(u,0,L,0) = 0$; indeed, u can be found close to the real even solution of $\tilde{F}(u,0,L+y,0) = 0$, where y is related to α by the relation

$$y = -\frac{1}{4}\ln\bigl(4\beta - 2\bigr) = -\frac{1}{4}\ln\bigl(4e^{2L}\alpha - 2\bigr).$$

In the course of the proof of skew stabilization, we will discover the following spatial scales in the problem: we have already 1 and L; the third scale is $\ln L$, which is the order of magnitude of the critical y for which the skew stabilization occurs.

We can now explain the organization of the paper; since we are in an almost scalar situation, we let $\alpha = \sqrt{\varepsilon}$ and we define

$$G(\xi,\eta,\tau,\varepsilon,\nu) = \Re F(\xi + i\sqrt{\varepsilon}\eta, \sqrt{\varepsilon}\tau, \sqrt{\varepsilon}, \nu) + \frac{i}{\sqrt{\varepsilon}}\Im F(\xi + i\sqrt{\varepsilon}\eta, \sqrt{\varepsilon}\tau, \sqrt{\varepsilon}, \nu).$$

In our continuation analysis, we need a starting point (u, ω) such that

$$F(u, \omega, 0, \nu) = 0. \tag{1.20}$$

In section 2, we prove that a solution u of (1.20) exists and decays exponentially at infinity only if m belongs to the interval $(0, 3/16)$. This solution is, up to phase and space translations, the unique positive even solution of

$$-mr'' + mr - r^3 + r^5 = 0,$$

as is proved at Lemma 2.1.

Then, we proceed to study precisely this $r = r(x, \nu)$, which has an explicit expression; however, we use mostly the asymptotic for $r(x, \nu)$, when ν is small (Lemma 2.2). Thus, we see that the shelf of r has indeed a half-length of L defined by (1.6). We study the linearized operator A around r; it is an unbounded self-adjoint operator in $L^2(\mathbb{R})$ given by

$$D(A) = H^2(\mathbb{R}), \quad Au = -mu'' + (m - 3r^2 + 5r^4)u.$$

We prove that the spectrum of A contains a group of eigenvalues $\{\lambda, 0\}$, and that

$$\lambda \sim -\frac{3\nu}{2}$$

is the lowest eigenvalue of A; moreover, this group of eigenvalues is bounded away from the remainder of the spectrum, uniformly in ν. The eigenvalue 0 corresponds to the translation mode, with eigenfunction r', and the eigenvalue λ corresponds to the shrinking-swelling mode, with eigenfunction s. This analysis is made possible by the following fact: if

$$\sigma(x, \nu) = -4\nu \frac{\partial r(x, \nu)}{\partial \nu} = \frac{\partial}{\partial L} r(x, \nu),$$

then, with an appropriate normalization of s,

$$|\sigma - s|_{H^2(\mathbb{R})} = O(\nu\sqrt{L}),$$

as is proved at Theorem 2.8.

The existence will be proved using the scaled equation

$$G(\xi, \eta, \tau, \varepsilon, \nu) = 0.$$

The scaled equation is more interesting from the point of view of continuation, because for $\alpha = 0$, $\omega = 0$, and it is easy to check that

$$\Im F(r, 0, 0, \nu) = 0,$$

which does not provide any information. However,

$$\Re G(\xi, \eta, \tau, 0, \nu) = \Re F(\xi, 0, 0, \nu)$$

so that r satisfies

$$\Re G(r, \eta, \tau, 0, \nu) = 0,$$

and η and τ are yet undetermined. The second equation is

$$\Im G((r, \eta, \tau, 0, \nu) = \tau r - r^3 - m\eta'' + m\eta - r^2\eta + r^4\eta = 0, \tag{1.21}$$

and it is studied in details in Section 3. Let θ and q denote the values of η and τ which satisfy (1.21). It is natural to study the operator B in $L^2(\mathbb{R})$ defined by

$$D(B) = H^2(\mathbb{R}), \quad Bu = -mu'' + (m - r^2 + r^4)u.$$

We can see immediately that 0 is the lowest eigenvalue of B and the corresponding eigenvector is r; this is not surprising, since it is the analytical translation of the S^1 equivariance of (1.7). In other words, ir is the phase rotation mode. Therefore, (1.21) will have a solution if and only if r is orthogonal to $r^3 - \theta r$; this determines θ; if we impose that q be orthogonal to r, it is uniquely determined.

For later purposes, we need an asymptotic on the second eigenvalue μ_2 of B; it is proved at Theorem 3.4 that $\mu_2 \sim C/L^2$, where C is a positive constant.

Section 4 is devoted to the existence proof. Preliminary computations showed that continuation is not good enough to obtain a satisfactory range of existence; what is needed is an ansatz for the pulse; it is obtained by taking $\nu^\flat = \nu e^{-4y}$, where y is some positive number bounded by L^p; the corresponding r, θ and q are denoted by $r^\flat$, $\theta^\flat$ and $q^\flat$. Now, $\varepsilon^\flat$ has to be determined: this is a version of the Lyapunov-Schmidt method of bifurcation theory. Our choice is to require that $\Re G(r^\flat, q^\flat, \theta^\flat, \varepsilon^\flat, \nu)$ is orthogonal to $s^\flat$, the shrinking-swelling mode corresponding to $\nu^\flat$. An asymptotic for $\varepsilon^\flat$ is given by

$$\varepsilon^\flat \sim \frac{\kappa}{4},$$

where

$$\kappa = (\nu - \nu^\flat)/(1 - \nu^\flat).$$

Let $U^\flat$ be the vector of components $(r^\flat, q^\flat, \theta^\flat, \varepsilon^\flat)$. The idea is to observe that this $U^\flat$ is almost a solution of $G(U^\flat, \nu) = 0$. Existence is proved using a version of the implicit function theorem with estimates, proved in Section 7 . The pulse obtained this way is denoted by u.

In other words, we approximate the pulse at $\alpha = \sqrt{\varepsilon^\flat}$ and ν by the pulse at $\alpha = 0$ and $\nu^\flat$.

Let us denote by $\mathcal{O}(1)$ any quantity bounded by a finite power of L.

The main result of this article, i.e. the proof of stabilization (section 5) uses the details of the proof of existence. If $(1-\kappa)\mathcal{D}$ is the differential of F with respect to u at (u, ω, α, ν), then,

$$\mathcal{D} = \mathcal{A}^\flat + \sqrt{\kappa}\,\mathcal{B} + \kappa\mathcal{C}.$$

Here,

$$\mathcal{A}^\flat = \begin{pmatrix} A^\flat & 0 \\ 0 & B^\flat \end{pmatrix},$$

where $A^\flat$ (resp. $B^\flat$) is A (resp. B) at $\nu^\flat$ instead of ν, and $\mathcal{B}$, $\mathcal{C}$ are 2×2 matrix of multiplication operators C_{ij}, $1 \le i, j \le 2$ such that

$$\mathcal{B} = \begin{pmatrix} 0 & \mathcal{B}_{12} \\ \mathcal{B}_{21} & 0 \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} \mathcal{C}_{11} & 0 \\ 0 & \mathcal{C}_{22} \end{pmatrix},$$
$$\|\mathcal{B}_{12}\|_{L^\infty} + \|\mathcal{B}_{21}\|_{L^\infty} + \|\mathcal{C}_{11}\|_{L^\infty} + \|\mathcal{C}_{22}\|_{L^\infty} = \mathcal{O}(1).$$

The idea is to consider the restriction of $\mathcal{D}$ to the generalized eigenspace corresponding to the eigenvalues of $\mathcal{B}$ which are close to zero. This eigenspace is of dimension 3; a basis of it is $\{s, iu, u'\}$, where iu spans the phase rotation mode, u' spans the space translation mode, and s corresponds to the shrinking-swelling mode. In this basis, the matrix of $\mathcal{B}$ is given by

$$\begin{pmatrix} M_{11} & 0 & 0 \\ M_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The sign of M_{11} determines the stability of the pulse: if $M_{11} > 0$, the pulse is stable (up to space and phase translation); if $M_{11} \le 0$, the pulse is unstable. Thus we have to give an asymptotic for M_{11}. For this purpose, we embed the operator $\mathcal{C}$ into an holomorphic family $\mathcal{D}(c)$ of operators depending on c, and we prove estimates using the strong properties of such families. In particular, we give a precise description of the expansion of a basis of eigenfunctions relative to the very small eigenvalues of $\mathcal{D}(c)$, and of the dual basis, and we validate these expansions.

With the residue theorem, we are able to describe M_{11} with sufficient precision, and the symmetries of the problem lead us to an almost completely explicit value for it (Lemma 5.4, Theorem 5.5 and Theorem 5.6). We conclude this asymptotic analysis at Proposition 5.8.

Thanks to a result of Henry [18], Chapter 5, Exercise 6, the linearized stability implies the following non linear stability result: take an initial condition for (1.7) which is equal to a pulse plus a small perturbation ; then, if $|\alpha| \geq \alpha_c$, the asymptotic state of the solution of (1.7) is a pulse possibly translated in space and in phase.

In section 6 we give the analogous asymptotic for the case when the μ_j's do not vanish; the appendix (section 7) gives an implicit function theorem with estimates; this theorem is the key to the existence result; in other words: our existence result is based on an ansatz: if the ansatz is good enough, then it is indeed a good approximation of the solution. When small parameters are involved, a correct argument deserves a proof.

A rather curious fact is that the number $\pi^2/6$ appears in the calculation of the expansion of M_{11}; in partial differential equations, it is usually related to a trace, but we have been unable to uncover such an origin; therefore, its presence may be a coincidence.

Some of these results were announced in [7] which contains a number of errors. A preprint [8] was circulated but never published as an article; the present article contains for the first time the approximate explanation of the skew-stabilization of the pulse and also the case of general coefficients as in (1.8).

There is considerable interest in the Ginzburg-Landau models; scanning the literature, one can find for example [36] which lists fluctuations in lasers, order-disorder transitions, population dynamics and ordering in uniaxial ferromagnetic films as domains where Ginzburg-Landau of the third degree has been used as a model. W. Eckhaus [9] states that Ginzburg-Landau of the third degree is "universal" for modulation equations, which is another way of saying that it behaves as a normal form.

The article [43] describes a large number of solutions of (1.1), perturbation expansions for large values of $\Im m_j$ and gives conjectures on the behavior of the solutions of (1.1) in different regions of the parameter space.

Ginzburg-Landau of the fifth order is much less generic than the third order Ginzburg-Landau. Its main merit is that it allows for subcritical bifurcation of the constant amplitude solutions.

Thual and Fauve explained in [41] the phenomenon they obtained in terms of the general picture of subcritical bifurcation, and also as a perturbation with respect to a nonlinear Schrödinger equation.

Shortly later, Malomed and Nepomnyashchy [30] considered the same equation (1.1) and explained the existence of a pulse by formal asymptotics. A careful examination of their results shows that they worked in a different range of parameters from ours.

Hakim, Jakobsen and Pomeau [16] have given a general idea of the bifurcation picture in a situation which is close to the present one; however, it is difficult to compare the situations, since their statements are not described with complete precision. One of their statements is the subcritical character of the bifurcation. While the bifurcation of space independent solutions is clearly subcritical, the bifurcation of the pulse is subcritical only by the fact that initially the solution is unstable, and it is stabilized along the solution curve; however, the typical picture of subcriticality as in Fig.1 of [41] has not been found in the present article.

Kapitula [22] gave a general theory for the existence of heteroclinic traveling wave solutions of the quintic Ginzburg-Landau equation with convective terms. Kapitula basically studies the persistence under perturbations of heteroclinic orbits close to

orbits which can be obtained almost explicitly when all coefficients are real. The proof relies on a very precise study of the perturbed invariant manifolds for a flow associated to the system. Possibly, his methods could be adapted to give existence of homoclinic solutions of (1.3).

According to a very striking phrase by Yves Meyer [31], the present work is like the success of the lock breaker: he/she has to use many pick locks, and the place looks messy; however, it is expected that once the door is open, and the lock can be dismantled and studied in details, someone will be able to devise a nice key which will open it in a single move.

Kapitula [22] used an entirely different set of techniques to devise a nice key for related existence questions, but a nice key to stability does not yet exist.

This article makes available a number of pick locks, to be stored in the tool box of the mathematical lock picker. It is more in the spirit of the SLEP method of [34] than in the spirit of the large literature on the analysis of the stability of traveling waves: solutions of the quintic Ginzburg-Landau equation have been analyzed in [23], [25], [27], [24], where the main difficulty is the bifurcation from the essential spectrum; the foundational work on the stability of traveling and standing wave solutions of semilinear parabolic equations is related to the Evans function; see in particular [19], [1] and [35], which were followed by a considerable literature, including in particular [12], [1], [20], [33], [26], [15]. The analysis of perturbation of periodic states has been taken up by the Evans function method in [13] [14] and by modulation equation methods in [40]. Solutions with several fronts or bumps have been constructed in [3], [4], [38], [37], [2] and [39].

It is possible that the methods of this article are close to those of [21], which however does not have the S^1 symmetry of the problem considered in the present article.

We would like to thank Stephan Fauve for introducing us to this problem, which revealed itself as infinitely more complicated than what we would have expected. The second author is glad to thank R.L. Pego, B. Malomed and A. Nepomnyashchy for fruitful discussions and exchanges of ideas.

Stéphane Descombes read the article in detail in the course of his first year of graduate studies, and spotted the defects, the typos and the errors. His criticisms improved considerably the article and he deserves praise and thanks for his patience.

2. The scalar equation and the corresponding linearized operator

We study in details the solutions of the equation

$$u_t - u_{yy} + mu - |u|^2 u(1 + i\alpha) + |u|^4 u = 0$$

under the assumption $\alpha = 0$ and

$$m \in \mathbb{R} \setminus 0. \tag{2.1}$$

We look for solutions of the form

$$u(y,t) = v(y)e^{i\omega t}.$$

If we substitute this expression into our equation, we can see that v and ω satisfy

$$i\omega v - v_{yy} + mv - |v|^2 v + |v|^4 v = 0. \tag{2.2}$$

Our first and elementary results on this case are summarized in the following lemma:

Lemma 2.1. *Let $v \in L^5_{loc}(\mathbb{R};\mathbb{C})$ solve (2.2) in the sense of distributions; if v does not vanish identically, then the following assertions hold:*

(i) *v is infinitely differentiable.*

(ii) *if $v(y)$ tends to zero as $|y|$ tends to $+\infty$, then ω vanishes.*

(iii) *under the assumption of (ii), the argument of v does not depend on y.*
(iv) *under the assumption of (ii), the number m is strictly positive.*
(v) *under the assumption of (ii), let $x = y\sqrt{m}$ and*

$$|v(y)| = r(x);$$

then, up to translation in space, r is the unique even positive solution of the ordinary differential equation

$$-mr'' + mr - r^3 + r^5 = 0 \tag{2.3}$$

which vanishes at infinity. In particular, the number m belongs to the interval $(0, 3/16)$.

Proof. (i) If v belongs to $L^5_{\mathrm{loc}}(\mathbb{R};\mathbb{C})$, then v'' belongs to $L^1_{\mathrm{loc}}(\mathbb{R};\mathbb{C})$, so that v' is locally absolutely continuous, and v is a function of class C^1 on $\mathbb{R}$, with values in $\mathbb{C}$. By an obvious induction argument, v is infinitely differentiable.

(ii) Equation (2.2) implies that v'' tends to 0 at infinity. Therefore, v' tends to zero at ∞: by Taylor's formula,

$$v'(y) = v(y+1) - v(y) - \int_0^1 v''(y+s)(1-s)\, ds;$$

the right hand side of this relation tends to 0 at infinity; therefore, the left hand side must also tend to 0.

Assume $\omega \neq 0$. System (2.2) is equivalent to a system of four ordinary differential equations of the first order in $\mathbb{R}^4$, and $0 \in \mathbb{R}^4$ is a critical point of this system. We apply the theory of invariant manifolds in a neighborhood of 0; the matrix of the linearized system at that point has two double eigenvalues $\pm\zeta$, where $\zeta^2 = m + i\omega$. We make the convention that $\Re\zeta > 0$. Therefore, $|v|$, $|v'|$ and $|v''|$ decrease exponentially fast at infinity. Multiply (2.2) by $\bar{v}$, and integrate over $\mathbb{R}$; we obtain after an integration by parts

$$i\omega \int |v|^2\, dy + \int |v'|^2\, dx + \int \left(m|v|^2 - |v|^4 + |v|^6\right) dy = 0.$$

This shows that ω vanishes.

(iii) Define

$$W(y) = m - |v(y)|^2 + |v(y)|^4.$$

We let $v = v_1 + iv_2$; then v_1 and v_2 solve the linear differential equation

$$w'' = Ww, \tag{2.4}$$

and they cannot both vanish identically. If w is a solution of (2.4) which does not vanish identically, then w and w' cannot vanish simultaneously; if w_1 and w_2 are two solutions of (2.4) which vanish at infinity, their Wronskian $w_1'w_2 - w_2'w_1$ is constant and vanishes at infinity; therefore, any two solutions of (2.4) are proportional. Thus v_1 and v_2 are linearly dependent, which proves the desired assertion.

(iv) Thanks to (iii), if v is a solution of (2.2), with $\omega = 0$, we may assume that it is real without loss of generality. Thus, it solves (2.4); moreover, v and v' cannot vanish simultaneously by uniqueness of solution of the Cauchy problem for (2.2). Therefore, we can find a continuous determination of the angle θ such that

$$w = r\cos\theta, w' = r\sin\theta. \tag{2.5}$$

It is immediate that this determination is also of class C^∞. Let us multiply (2.2) by $2v'$, and integrate, remembering that $\omega = 0$ and v is real; we obtain

$$-|v'|^2 + m|v|^2 - |v|^4/2 + |v|^6/3 = \text{constant}. \tag{2.6}$$

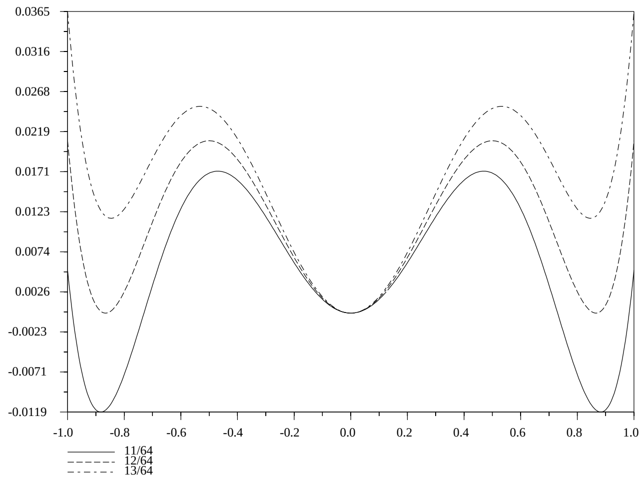

FIGURE 2. The graph of Φ for $m = 11/64, m = 12/64, m = 13/64$.

By taking the limit of the left hand side of (2.6), we see that the constant on the right hand side of (2.6) vanishes. We use (2.5), and we can see after substituting into (2.6) and dividing by r^2 that

$$-\sin^2\theta + m\cos^2\theta = \frac{r^2}{2} - \frac{r^4}{3}.$$

Assume that m is strictly negative. As x tends to infinity, the upper limit of the left hand side of the above equation is at most equal to $-\min\bigl(1, |m|\bigr)$, while the right hand side of this equation tends to 0; thus, we have a contradiction.

(v) The proof of (iv) shows that v can be taken real; define a function r by

$$v(y) = zr(y\sqrt{m});$$

then r is a real solution of

$$-mr'' + mr - r^3 + r^5 = 0$$

which decays at infinity. Moreover, if we define

$$\Phi(r) = mr^2 - \frac{r^4}{2} + \frac{r^6}{3},$$

(2.6) implies

$$m|r'|^2 = \Phi(r).$$

In order to have a non trivial solution of (2.3) which tends to 0 at $\pm\infty$, we must choose m so that Φ can vanish for non zero values of r; letting

$$\Psi(X) = m - \frac{X}{2} + \frac{X^2}{3} = X^{-1}\Phi(\sqrt{X}),$$

it is immediate that ψ can vanish for positive values of X if and only if

$$m < \frac{3}{16}.$$

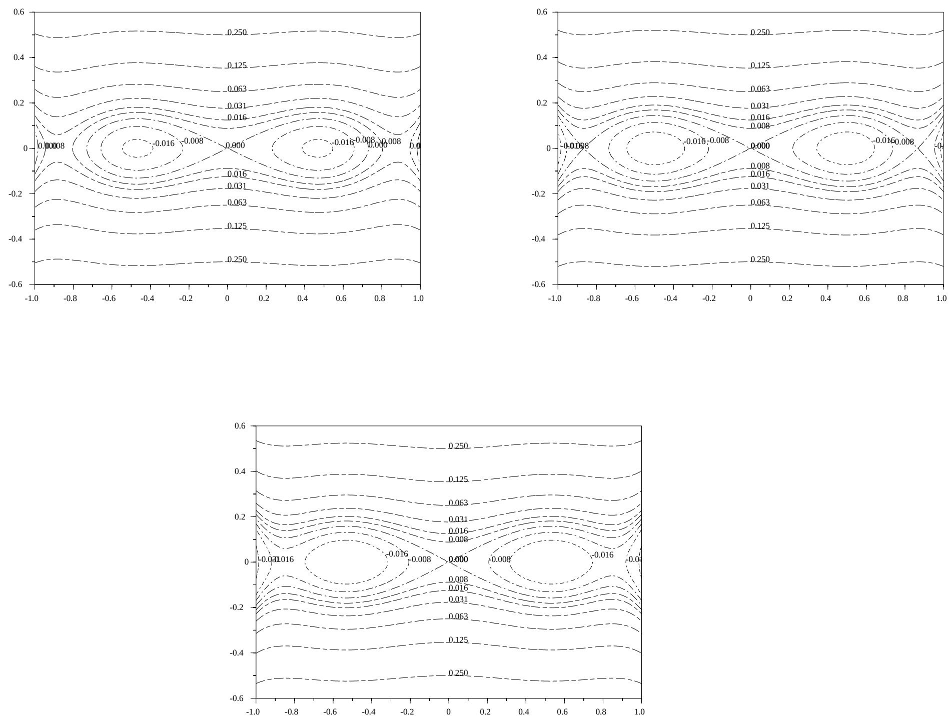

FIGURE 3. The level lines of $(r,s) \mapsto s^2 - \Phi(r)$ for $m = 11/64$, $m = 12/64$, $m = 13/64$.

Henceforth, we will write

$$m = \frac{3}{16}(1-\nu) \Longleftrightarrow \nu = 1 - \frac{16m}{3}.$$

The level curves of $(r,s) \mapsto s^2 - \Phi(r)$ are represented at Fig. 3; this figure shows that there exist exactly two homoclinic orbits of (2.3) through 0 for $m < 3/16$; on one of them, r takes only positive values, and on the other one r takes only negative values. Therefore, we choose the homoclinic orbit which is situated in the half plane $r \geq 0$. The solution r is still defined only up to translation. When r reaches a maximum at some point x_0, $r'(x_0) = 0$, and $\psi(r^2(x_0))$ vanishes.Therefore,

$$r^2(x_0) = \frac{3(1-\sqrt{\nu})}{4}. \tag{2.7}$$

Since (2.3) is invariant by the reflexion $x \mapsto 2x_0 - x$, we can see that $r(x) = r(2x_0 - x)$; if u has more than one maximum, it is periodic, which is possible only if u vanishes identically. Therefore, r has exactly one maximum.

If we choose the translation parameter so that r attains its maximum at 0, then r is even and positive over $\mathbb{R}$, which concludes the proof of the lemma. $\square$

In what follows, the dependence of r or $R = r^2$ over ν will be emphasized from time to time, in which case, we will write $r(x,\nu)$ or $R(x,\nu)$ instead of $r(x)$ or $R(x)$.

The value of R can be found explicitly; it is equal to

$$R(x) = \frac{3}{4}\frac{1-\nu}{1+\sqrt{\nu}\cosh 2x}.$$

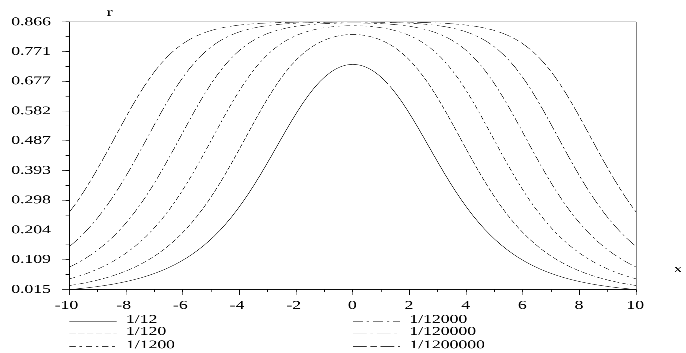

FIGURE 4. The graph of R over the interval $[-10, 10]$, for $\nu = 10^{-p}/12, 0 \leq p \leq 5$.

If we draw a graph of the values of r or R for different values of ν, we can observe that the 'width' of r or R increases logarithmically with $1/\nu$; see 4.

In fact, as ν decreases to 0 R is very close to a two front solution, and the distance between these two fronts can be estimated very precisely. This important observation in stated and proved in next Lemma:

Lemma 2.2. *Let*

$$L = \frac{1}{4} \ln \frac{4}{\nu} \iff \nu = 4e^{-4L}, \tag{2.8}$$

and

$$\tilde{R}(x) = \frac{3}{4} \frac{1}{1 + e^{2x}}. \tag{2.9}$$

Then, we have the estimates for all $x \in \mathbb{R}$

$$\left| R(x + L) - \tilde{R}(x) \right| \leq \nu e^{-2x}. \tag{2.10}$$

$$\left| R'(x + L) - \tilde{R}'(x) \right| \leq O(1)\nu e^{-2x}. \tag{2.11}$$

Moreover, $\tilde{r} = \sqrt{\tilde{R}}$ is a solution of the differential equations

$$-\frac{3}{16}\tilde{r}'' + \tilde{W}\tilde{r} = 0 \tag{2.12}$$

where

$$\tilde{W} = \frac{3}{16} - \tilde{R} + \tilde{R}^2, \tag{2.13}$$

and

$$\tilde{r}' = -\tilde{r}\left(1 - \frac{4\tilde{r}^2}{3}\right). \tag{2.14}$$

Proof. We infer from (2.9) the relation

$$e^{-2x} = \frac{\tilde{R}(x)}{3/4 - \tilde{R}(x)}; \tag{2.15}$$

it is also immediate that $\tilde{R}$ satisfies the ordinary differential equation

$$\tilde{R}' = -2\tilde{R}(1 - 4\tilde{R}/3), \tag{2.16}$$

which can also be found by passing to the limit as ν tends to 0 in the differential equation satisfied by R:

$$R' = \pm 2R\sqrt{\psi(R)/m}. \tag{2.17}$$

The definition (2.8) of L and a direct computation give

$$R(\cdot + L) = \frac{(1-\nu)\tilde{R}(1-4\tilde{R}/3)}{1-4\tilde{R}/3+4\nu\tilde{R}^2/9}, \tag{2.18}$$

from which we infer the identity

$$R(\cdot + L) - \tilde{R} = -\frac{\nu\tilde{R}(1-2\tilde{R}/3)^2}{1-4\tilde{R}/3+4\nu\tilde{R}^2/9}. \tag{2.19}$$

Thanks to (2.15), we deduce immediately (2.10) from (2.19). If we differentiate (2.19) with respect to x, we find

$$\begin{aligned} &R'(\cdot + L) - \tilde{R}' \\ &= -\nu\tilde{R}'\left[(-4/3+8\nu\tilde{R}/9)\frac{\tilde{R}(1-2\tilde{R}/3)^2}{(1-4\tilde{R}/3+4\nu\tilde{R}^2/9)^2} - \frac{(1-8\tilde{R}/3+4\tilde{R}^2/3}{1-4\tilde{R}/3+4\nu\tilde{R}^2/9}\right]. \end{aligned}$$

Then, with the help of (2.16) and (2.15), relation (2.11) is clear. By a passage to the limit as ν tends to 0, or by a direct computation, we see that (2.12), (2.13) and (2.14) hold. As x tends to $+\infty$, $\tilde{R}$ tends to 0 and V tends to $3/16$. $\square$

The linearized operator around r is an unbounded operator A in $L^2(\mathbb{R})$ defined by

$$D(A) = H^2(\mathbb{R}), \quad Au = -mu'' + Vu, \quad V = m - 3r^2 + 5r^4.$$

The spectral properties of A are important for what follows:

Lemma 2.3. *The operator A is self-adjoint; its continuous spectrum is included in the interval $[m, +\infty)$; the infimum of the spectrum of A is a strictly negative number λ, to which corresponds an even strictly positive eigenfunction s. The function r' is an eigenfunction of A corresponding to the eigenvalue 0. Both λ and 0 are simple eigenvalues.*

Proof. Clearly, A is self-adjoint. We have

$$V(\pm\infty) = m > 0;$$

The essential spectrum of A is contained in $[m, +\infty)$ (see Kato, [28],Theorem 5.26), and the spectrum of A is bounded from below by $\min_x V(x)$; for all $\beta > 0$, the intersection of the spectrum of A with $(-\infty, m-\beta]$ contains only eigenvalues of finite multiplicity; since A is a Schrödinger operator in one-dimensional space, these eigenvalues are all simple; moreover, since r is even, V is even, and the eigenfunctions of A are even or odd; in particular, an eigenfunction corresponding to the lowest eigenvalue λ of A does not change sign, thanks to the maximum principle; therefore, it is even, since such an eigenfunction cannot vanish together with its derivative.

Denote by s an eigenfunction of A corresponding to λ; this s can be chosen so that

$$s(x) > 0, \forall x \in \mathbb{R}.$$

Differentiating (2.3) with respect to x, we obtain

$$Ar' = 0.$$

It is clear that r' belongs to $L^2(\mathbb{R})$; therefore, it is an eigenvector of A, corresponding to the eigenvalue 0. On the other hand, r' is odd; therefore, we have proved that the infimum of the spectrum of A is a strictly negative number, λ. $\square$

We will obtain later much more precise information on the spectrum of A; but this information is easier to obtain if we take into account the nonlinearities of our problem ; in particular, we will asymptotically describe the eigenfunction s in terms of the derivative of r with respect to ν, a feature which is clearly specific to our nonlinear situation. Thus, we have to study the dependence with respect to ν of a number of objects which appear in this section.

Lemma 2.4. (i) *The function*

$$\begin{array}{ccc}]0,1[& \to & H^2(\mathbb{R}) \\ \nu & \mapsto & r(x,\nu) \end{array}$$

is infinitely differentiable.

(ii) *The eigenvalue $\lambda(\nu)$ and the eigenprojection $P(\nu)$ on $\mathbb{R}s(\nu)$ are infinitely differentiable, respectively with values in $\mathbb{R}$ and in the space of continuous operators in $L^2(\mathbb{R})$.*

Proof. (i) Denote by $H^2_{\text{even}}(\mathbb{R})$ (resp. $L^2_{\text{even}}(\mathbb{R})$) the subspace of even functions belonging to $H^2(\mathbb{R})$ (resp. $L^2(\mathbb{R})$). Define a mapping f from $H^2_{\text{even}}(\mathbb{R}) \times (0,1)$ to $L^2_{\text{even}}(\mathbb{R})$ by

$$f(v,\nu) = -mv'' + mv - v^3 + v^5.$$

This mapping is of class C^∞, we have $f(r,\nu) = 0$, and $D_1 f(r,\nu) = A$, which is an isomorphism from H^2_{even} to $L^2_{\text{even}}(\mathbb{R})$. Therefore, the implicit function theorem applies, and r is a C^∞ function of ν with values in $H^2_{\text{even}}(\mathbb{R})$.

(ii)If we divide A by $1-\nu$ we obtain a Schrödinger operator whose differential part is constant, and whose potential part depends in a C^∞ fashion on ν, according to part (i) of this proof; therefore, we can apply the results of Kato, [28] IV.3, which enable us to conclude. $\square$

Define three new functions $\sigma(x,\nu)$ and $\sigma_2(x,\nu)$ by

$$\begin{aligned} S(x,\nu) &= -4\nu\frac{\partial R}{\partial \nu} = \frac{\partial}{\partial L}R\big(x, 4e^{-4L}\big) \\ \sigma(x,\nu) &= -4\nu\frac{\partial r(x,\nu)}{\partial \nu} = \frac{\partial}{\partial L}r(x,4e^{-4L}), \\ \sigma_2(x,\nu) &= -4\nu\frac{\partial \sigma(x,\nu)}{\partial \nu} = \frac{\partial^2}{\partial L^2}r(x,4e^{-4L}). \end{aligned} \tag{2.20}$$

The following lemma states that $S+R_x$, $\sigma+r_x$ and $\sigma_2+2\sigma_x+r_{xx}$ are very close to 0; this is really a consequence of the non linearity; the benefit will be even greater when we will show below that σ is an excellent approximation of the eigenfunction of A corresponding to its lowest eigenvalue.

Lemma 2.5. *We have the following estimates for all $x \in \mathbb{R}^+$, and all $\nu \in (0,1)$:*

$$\left|\frac{S(x,\nu)+R_x(x,\nu)}{R(x,\nu)}\right| \le \frac{O(1)\nu}{1-4\tilde{R}(x-L)/3}, \tag{2.21}$$

$$\left|\sigma(x,\nu) + \frac{\partial r}{\partial x}(x,\nu)\right| \le O(1)\sqrt{\nu}\, e^{-x}, \tag{2.22}$$

and

$$\left|\sigma_2(x,\nu) + 2\frac{\partial \sigma}{\partial x}(x,\nu) + \frac{\partial^2 r}{\partial x^2}(x,\nu)\right| \le O(1)\sqrt{\nu}\, e^{-x}, \tag{2.23}$$

Proof. The differentiation of (2.18) with respect to L gives

$$\frac{1}{R\big(\cdot+L,4e^{-4L}\big)}\frac{1}{\partial L}R\big(\cdot+L,4e^{-4L}\big)=-\frac{4\nu}{1-\nu}\frac{(1-2\tilde{R}/3)^2}{1-4\tilde{R}/3+4\nu\tilde{R}^2/9},$$

which implies immediately (2.21). An analogous computation implies

$$\frac{\partial}{\partial L}\big[r(x+L,4e^{-4L})\big]=r_x(x+L,4e^{-4L})+\sigma(x+L,4e^{-4L}), \tag{2.24}$$

and

$$\begin{aligned}\frac{\partial^2}{\partial L^2}\big[r(x+L,4e^{-4L})\big]&=r_{xx}(x+L,4e^{-4L})\\&\quad+2\sigma_x(x+L,4e^{-4L})+\sigma_2(x+L,4e^{-4L}).\end{aligned} \tag{2.25}$$

Differentiating (2.18), we find

$$\frac{\partial}{\partial L}r(\cdot+L,4e^{-4L})=\frac{2\nu\tilde{r}(1-4\tilde{R}/3)(1-2\tilde{R}/3)^2}{(1-4\tilde{R}/3+4\nu\tilde{R}^2/9)^2},$$

and

$$\begin{aligned}\frac{\partial^2}{\partial L^2}r(\cdot+L,4e^{-4L})=&-\frac{8\nu\tilde{r}(1-4\tilde{R}/3)(1-2\tilde{R}/3)^2(1-4\tilde{R}/3-4\nu\tilde{R}^2/9)}{(1-4\tilde{R}/3+4\nu\tilde{R}^2/9)^3}\\&-\frac{4\nu^2\tilde{r}(1-4\tilde{R}/3)^2(1-2\tilde{R}/3)^4}{(1-4\tilde{R}/3+4\nu\tilde{R}^2/9)^4}.\end{aligned}$$

Observe that

$$\frac{\nu\tilde{r}}{1-4\tilde{R}/3}=\nu\sqrt{\frac{3}{4}}e^{-x}\big(1+e^{-2x}\big)^{1/2},$$

so that:

$$\forall x\geq -L,\quad \frac{\nu\tilde{r}}{1-4\tilde{R}/3}\leq O(1)\sqrt{\nu}e^{-x-L}; \tag{2.26}$$

moreover we have also

$$\forall x\geq -L,\quad \frac{\nu}{1-4\tilde{R}/3}\leq O(1); \tag{2.27}$$

We use the information given by (2.26) and (2.27), to infer that

$$\begin{aligned}\forall x\geq -L,\quad &\left|\frac{\partial}{\partial L}r(\cdot+L,4e^{-4L})\right|\leq O(1)\sqrt{\nu}e^{-x-L},\\&\left|\frac{\partial^2}{\partial L^2}r(\cdot+L,4e^{-4L})\right|\leq O(1)\sqrt{\nu}e^{-x-L};\end{aligned}$$

the conclusion of the lemma follows immediately. □

For later purposes, we define

$$S(x+L,4e^{-4L})=\frac{\partial}{\partial L}R(x+L,4e^{-4L}).$$

From here, we will obtain an upper estimate for the lower bound of the spectrum of A:

Lemma 2.6. *We have*

$$\frac{(A\sigma,\sigma)}{(\sigma,\sigma)}\sim-\frac{3\nu}{2}.$$

Proof. Let us compute $A\sigma$ by differentiating the equation

$$-mr'' + mr - r^3 + r^5 = 0$$

with respect to L; we obtain

$$A\sigma = \rho \tag{2.28}$$

where

$$\rho = \frac{4\nu}{1-\nu}(r^5 - r^3). \tag{2.29}$$

To prove our assertion, we will have to calculate (σ, σ) and (σ, ρ). From Lemma 2.5, we can see that

$$\int \sigma^2(x,\nu)\,dx = 2\int_0^\infty r_x(x,\nu)^2\,dx + O(\sqrt{\nu}),$$

and from estimate (2.11), we have

$$2\int_0^\infty r_x(x,\nu)^2\,dx = 2\int_{-L}^\infty \tilde{r}'(x)^2\,dx + O(\sqrt{\nu});$$

but relations (2.14) and (2.16) imply that

$$|\tilde{r}'|^2 = -\frac{1}{2}\tilde{R}'\left(1 - \frac{4\tilde{R}}{3}\right);$$

we infer from relation (2.7) that

$$2\int_{-L}^\infty |\tilde{r}'|^2\;dx = \frac{3}{8} + O\big(\sqrt{\nu}\big), \tag{2.30}$$

so that

$$(\sigma, \sigma) = \frac{3}{8} + O(\sqrt{\nu}). \tag{2.31}$$

On the other hand,

$$\begin{aligned} (r^5 - r^3, \sigma) &= -2\int_0^\infty (r^5 - r^3) r_x\,dx + O(\sqrt{\nu}) \\ &= 2\left(\frac{R(0,\nu)^3}{6} - \frac{R(0,\nu)^2}{4}\right) + O(\sqrt{\nu}) \\ &= -\frac{9}{64} + O(\sqrt{\nu}). \end{aligned} \tag{2.32}$$

Thus, we obtain from (2.29) and (2.32) the relation

$$(A\sigma, \sigma) = -\frac{9\nu}{16} + O(\nu^{3/2}). \tag{2.33}$$

The conclusion of the lemma is a direct consequence of (2.31) and (2.33). □

Theorem 2.7. *There exists a constant K such that for all small enough ν the spectrum of A is partitioned in its intersection with the interval $[-K, K]$ and its intersection with interval $[2K, +\infty)$.*

Proof. Define

$$\tilde{V}(x) = \frac{3}{16} - 3\tilde{R} + 5\tilde{R}^2.$$

It is clear that $\tilde{r}'$ satisfies the Schrödinger equation

$$-\tilde{r}''' + \tilde{V}\tilde{r}' = 0.$$

As $\tilde{r}'$ is strictly negative for all x, this means that 0 is the lower bound of the spectrum of the operator $\tilde{A}$ defined by

$$D(\tilde{A}) = H^2(\mathbb{R}), \tilde{A}v = -v'' + \tilde{V}v.$$

As $\tilde{V}(-\infty) = 3/4$ and $\tilde{V}(+\infty) = 3/16$, the essential spectrum of $\tilde{A}$ is included in $[3/16, +\infty)$, and thus the second eigenvalue of $\tilde{A}$ is some number $\Lambda > 0$.

We use now the twisting trick of E. B. Davies [6]: define the operators

$$\tilde{\mathcal{J}} = \begin{pmatrix} -\partial^2 + \tilde{V}(\cdot - L) & 0 \\ 0 & -\partial^2 + \tilde{V}(-\cdot - L) \end{pmatrix}, \quad \mathcal{J} = \begin{pmatrix} -\partial^2 + 3/4 & 0 \\ 0 & A \end{pmatrix};$$

let η be an infinitely differentiable function from $\mathbb{R}$ to $[0, \pi/2]$ which vanishes for $x \leq -1$ and is equal to $\pi/2$ for $x \geq 1$ and let $\eta_L = \eta(\cdot/L)$. Define now a unitary transformation in $L^2(\mathbb{R})^2$ by its matrix

$$U_L = \begin{pmatrix} \cos\eta_L & -\sin\eta_L \\ \sin\eta_L & \cos\eta_L \end{pmatrix}.$$

We compute $U_L \mathcal{J} U_L^*$ and we find that

$$U_L \frac{\partial^2}{\partial x^2} U_L^* = \frac{\partial^2}{\partial x^2} + -\frac{\eta'(\cdot/L)}{L} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial x} - \frac{\eta''(\cdot/L)}{L^2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \frac{\big(\eta'(\cdot/L)\big)^2}{L^2}$$

and

$$U_L \begin{pmatrix} V & 0 \\ 0 & 3/4 \end{pmatrix} U_L^* = \begin{pmatrix} V\cos^2\eta_L + \big(3\sin^2\eta_L\big)/4 & \big(\cos\eta_L \sin\eta_L\big)(V - 3/4) \\ \big(\cos\eta_L \sin\eta_L\big)(V - 3/4) & V\sin^2\eta_L + \big(3\cos^2\eta_L\big)/4 \end{pmatrix}.$$

An elementary calculation shows now that

$$U_L \mathcal{J} U_L^* = \tilde{\mathcal{J}} + B_L \frac{\partial}{\partial x} + C_L$$

where B_L and C_L are matrix valued functions whose norm tends to 0 as L tends to infinity. It is plain that the resolvent sets of $\tilde{\mathcal{J}}$ and $\tilde{A}$ are identical.

If E and F are Banach spaces, $\mathcal{L}(E, F)$ is the space of linear bounded operators from E to F and $\| \, \|_{\mathcal{L}(E,F)}$ the corresponding norm; if $E = F$, we abbreviate $\mathcal{L}(E, F)$ to $\mathcal{L}(E)$. If E is the space $L^2(\mathbb{R})^n$ we abbreviate $\mathcal{L}(E)$ to $\mathcal{L}$.

Let z belong to the resolvent set of $\tilde{A}$; then for every g in $L^2(\mathbb{R})^2$, $(\tilde{\mathcal{J}} - z)^{-1}$ belongs to $H^2(\mathbb{R})^2$, and the function $z \mapsto \|(\partial^2/\partial x^2)(\tilde{\mathcal{J}} - z)^{-1}\|_{\mathcal{L}(L^2)}$ is continuous on the resolvent set of $\tilde{\mathcal{J}}$.

It is equivalent to solve

$$\left(\tilde{\mathcal{J}} + B_L \frac{\partial}{\partial x} + C_L - z\right) u = f$$

and

$$g + \left(B_L \frac{\partial}{\partial x} + C_L\right) (\tilde{\mathcal{J}} - z)^{-1} g = f;$$

the previous considerations show that for all z in the resolvent set of $\tilde{\mathcal{J}}$, we can find $L(z)$ such that for all $L \geq L(z)$, z is also in the resolvent set of $\mathcal{J}$. In particular, if we take $K = \Lambda/3$, we can find $\bar{L}$ such that for all $L \geq \bar{L}$, the segment $(K, 2K)$ is included in the resolvent set of $\mathcal{J}$. In particular, we see also that for all $L \geq \bar{L}$, the generalized eigenspace of A relative to the interval $[-\min V, K]$ contains exactly two eigenvalues; the minimum of V is equal to $-21/80 + O(\nu)$; the above considerations show that these two eigenvalues tend to 0 as ν tends to 0. $\square$

Let us prove now that λ is very close to $(A\sigma,\sigma)/(\sigma,\sigma)$.

Denote by Π the projection onto the sum of the eigenspaces relative to 0 and λ. We have

$$\Pi = \frac{1}{2\pi i}\int_\gamma (\zeta - A)^{-1}\,d\zeta, \tag{2.34}$$

where γ is a circle of radius $3K/2$, which is traveled once in the positive direction.

Theorem 2.8. *We have the following estimates:*

$$|\sigma - \Pi\sigma|_{H^2(\mathbb{R})} = O(\nu\sqrt{L}), \tag{2.35}$$

and

$$\lambda = -\frac{3\nu}{2} + O\big(\nu^{3/2}\big). \tag{2.36}$$

In particular, we can choose $s = \Pi\sigma$ as an eigenvector of A relative to the eigenvalue λ, as soon as ν is small enough. Moreover, if $\hat\sigma$ is the unique solution of

$$\Pi\hat\sigma = 0, \quad A\hat\sigma = (\mathbf{1} - \Pi)(\lambda\sigma - \rho) \tag{2.37}$$

then

$$|\Pi\sigma - \sigma - \hat\sigma|_{H^2(\mathbb{R})} = O\big(\nu^2\sqrt{L}\,\big). \tag{2.38}$$

Proof. We observe that for $|\zeta| = 3K/2$, $\zeta - A$ is an isomorphism from $H^2(\mathbb{R})$ to $L^2(\mathbb{R})$ which transforms even functions into even functions, and odd functions into odd functions. Therefore, since σ is even by construction, $\Pi\sigma$ is even too.

We rewrite the identity $A\sigma = \rho$ as

$$\sigma = \frac{\rho}{\zeta} + \frac{(\zeta - A)\sigma}{\zeta}.$$

Hence, by integration along γ,

$$\Pi\sigma = \sigma + \frac{1}{2\pi i}\int_\gamma (\zeta - A)^{-1}\zeta^{-1}\rho\,d\zeta. \tag{2.39}$$

Since $|\rho| = O(\nu\sqrt{L})$, we obtain

$$|\sigma - \Pi\sigma| = O(\nu\sqrt{L}), \tag{2.40}$$

If ν is small enough, (2.40) implies that $\Pi\sigma$ does not vanish identically. Thus, by construction it is an eigenfunction of A relative to the eigenvalue 0 or to the eigenvalue λ. But we know that $\Pi\sigma$ is even; thus, it has to be an eigenfunction of A relative to λ, and

$$\lambda = \frac{(A\Pi\sigma,\Pi\sigma)}{(\Pi\sigma,\Pi\sigma)}.$$

But

$$(A\Pi\sigma,\Pi\sigma) = (\rho,\Pi\sigma) = (\rho,\sigma) + (\rho,\Pi\sigma - \sigma) = (\rho,\sigma) + O(L\nu^2).$$

and, on the other hand

$$(\Pi\sigma,\Pi\sigma) = (\sigma,\Pi\sigma) = (\sigma,\sigma) + (\Pi\sigma - \sigma,\sigma) = (\sigma,\sigma) + O(\nu\sqrt{L}).$$

This proves (2.36).

In order to obtain (2.35) which holds in H^2 norm, while (2.35) holds in L^2 norm, we subtract the relation $A\Pi\sigma = \lambda\Pi\sigma$ from the relation $A\sigma = \rho$ and we obtain

$$m\big|\sigma_{xx} - (\Pi\sigma)_{xx}\big| = \big|\rho - \lambda\Pi\sigma - V(\sigma - \Pi\sigma)\big| = O(\nu\sqrt{L}).$$

For the last assertion of the theorem, we remark thanks to Cauchy's theorem, the last term in (2.39) can be rewritten as

$$\frac{1}{2\pi i}\int_\gamma (\zeta - A)^{-1}\zeta^{-1}\rho\, d\zeta = \frac{1}{2i\pi}\int_\gamma (\zeta - A)^{-1}\zeta^{-1}(\mathbf{1}-\Pi)\rho\, d\zeta = \tilde\sigma$$

and $\tilde\sigma$ is the unique solution of

$$\Pi\tilde\sigma = 0, \quad A\tilde\sigma = (\mathbf{1}-\Pi)\rho. \tag{2.41}$$

Subtracting (2.41) from (2.37), we find that

$$\Pi(\tilde\sigma - \hat\sigma) = 0, \quad A(\tilde\sigma - \hat\sigma) = \lambda\big(\sigma - \Pi\sigma\big).$$

and this implies immediately estimate (2.38), which concludes the proof. □

We need also the equation satisfied by σ_2. If we differentiate (2.28) with respect to L, we obtain

$$A\sigma_2 + (20r^3 - 6r)\sigma^2 = \rho_2, \tag{2.42}$$

where

$$\rho_2 = -4\nu\frac{\partial\rho}{\partial\nu} + \frac{3\nu}{4}(\sigma'' - \sigma) = -\frac{16\nu}{(1-\nu)^2}(r^5 - r^3) + \frac{8\nu}{1-\nu}(5r^4 - 3r^2)\sigma.$$

Therefore,

$$|\rho_2| = O\big(\sqrt{L}\,\nu\big). \tag{2.43}$$

Moreover an analogous and straightforward argument shows that

$$\frac{\partial\lambda}{\partial L} = 0(\nu), \quad \left|\frac{\partial s}{\partial L} - \sigma_2\right| = O\big(\sqrt{L}\,\nu\big). \tag{2.44}$$

The final information we obtain are asymptotics on $\partial\lambda/\partial L$ on $\partial s/\partial L$:

Lemma 2.9. *The following estimates hold:*

$$\frac{\partial\lambda}{\partial L} = 6\nu + O\big(\nu^{3/2}\big), \tag{2.45}$$

$$\left|\frac{\partial s}{\partial L} - \sigma_2\right|_{H^2(\mathbb{R})} = O\big(\nu\sqrt{L}\,\big). \tag{2.46}$$

Proof. For the first estimate, we differentiate the relation

$$A\Pi\sigma = \lambda\Pi\sigma$$

with respect to L; we multiply scalarly the result

$$\frac{\partial A}{\partial L}\Pi\sigma + A\frac{\partial\Pi\sigma}{\partial L} = \frac{\partial\lambda}{\partial L}\Pi\sigma + \lambda\frac{\partial\Pi\sigma}{\partial L}$$

by $\Pi\sigma$ and we obtain

$$|\Pi\sigma|^2\,\frac{\partial\Pi\sigma}{\partial L} = \left(\frac{\partial A}{\partial L}\Pi\sigma, \Pi\sigma\right).$$

Since

$$\frac{\partial A}{\partial L} = \frac{4\nu}{1-\nu}\big(A - 3r^2 + 5r^4\big) - 6r\sigma + 20r^2\sigma, \tag{2.47}$$

we may write

$$\left(\frac{\partial A}{\partial L}\Pi\sigma, \Pi\sigma\right) = 4\nu\big((3r^2 - r^4)\sigma, \sigma\big) + \big((20r^3 - 6r)\sigma^2, \sigma + 2\hat\sigma\big) + O(\nu^2 L).$$

Thanks to (2.42),

$$\begin{aligned}\big((20r^3-6r)\sigma^2,\sigma+2\hat\sigma\big)&=\big(A\sigma_2-\rho_2,\sigma\big)\\&=(\sigma_2,\rho)-(\rho_2,\sigma)+(\sigma_2,\lambda\sigma-\rho)+O\big(\nu^{3/2}\big);\end{aligned}$$

here we have used (2.37) and (2.35). Therefore,

$$\left(\frac{\partial A}{\partial L}\Pi\sigma,\Pi\sigma\right)=4\nu\big((3r^2-r^4)\sigma,\sigma\big)+2\lambda\sigma_2,\sigma)-\frac{\partial}{\partial L}(\rho,\sigma)+O(\nu^{3/2}).$$

At this point it is clear that there exists a number c such that

$$\frac{\partial\lambda}{\partial L}=c\nu+O\big(\nu^{3/2}\big),\tag{2.48}$$

but the explicit calculation of c is tedious. It can be avoided by integrating (2.48) with respect to L: we see that

$$\lambda=-c\nu/4+O\big(\nu^{3/2}\big),$$

and we infer from (2.36) that $c=6$.

The other asymptotics are obtained as follows: we differentiate the relation

$$\Pi\sigma=\int_\gamma(\zeta-A)^{-1}\sigma\,d\sigma$$

with respect to L:

$$\frac{\partial\Pi\sigma}{\partial L}=\frac{1}{2i\pi}\int_\gamma(\zeta-A)^{-1}\Big[\frac{\partial A}{\partial L}(\zeta-A)^{-1}\sigma+\sigma_2\Big]\,d\zeta.$$

We deduce from (2.38) that for $\zeta\in\gamma$

$$\big|(\zeta-A)^{-1}\sigma-(\zeta-\lambda)^{-1}\big|_{H^2(\mathbb{R})}=O\big(\nu\sqrt{L}\big);$$

therefore, it suffices to find an asymptotics for

$$\frac{1}{2i\pi}\int\gamma(\zeta-A)^{-1}(\zeta-\lambda)^{-1}\Big[\frac{4\nu}{1-\nu}\big(\rho+(3r^2-5r^4)\sigma+\rho_2-A\sigma_2+(\zeta-\lambda)\sigma_2\Big]\,d\zeta.$$

But

$$(\zeta-A)^{-1}(\zeta-\lambda)^{-1}\big(-A\sigma_2+(\zeta-A)\sigma_2\big)=(\zeta-\lambda)^{-1}-\lambda(\zeta-A)^{-1}(\zeta-\lambda)^{-1}$$

and it is clear now that (2.48) holds. □

3. The second equation

Let us first observe that we need only consider the case $\alpha>0$: the mapping $\alpha\mapsto-\alpha$ transforms a solution into its conjugate.

There are two different ways of writing the full problem, each with a different scaling. The unscaled problem is

$$-\frac{3}{16}(1-\nu)u''+\frac{3}{16}(1-\nu)u-(1+i\alpha)|u|^2u+|u|^4u+i\omega u=0.\tag{3.1}$$

Therefore, we define

$$F(u,\omega,\alpha,\nu)=-\frac{3}{16}(1-\nu)u''+\frac{3}{16}(1-\nu)u-(1+i\alpha)|u|^2u+|u|^4u+i\omega u.$$

Whenever necessary, we will use the components of u, denoting them by u_1 and u_2, so that $u=u_1+iu_2$, and we define similarly the components of F, F_1 and F_2; therefore, we have

$$F=F_1+iF_2.$$

Let $\alpha = \sqrt{\varepsilon}$, $u_1 = \xi$, $u_2 = \eta\sqrt{\varepsilon}$, $\omega = \tau\sqrt{\varepsilon}$. The scaled problem is

$$G(\xi, \eta, \tau, \varepsilon, \nu) = 0, \tag{3.2}$$

where we define

$$G(\xi, \eta, \tau, \varepsilon, \nu) = F_1(\xi + i\sqrt{\varepsilon}\,\eta, \sqrt{\varepsilon}\tau, \sqrt{\varepsilon}, \nu) + \frac{i}{\sqrt{\varepsilon}} F_2(\xi + i\sqrt{\varepsilon}\,\eta, \sqrt{\varepsilon}\tau, \sqrt{\varepsilon}, \nu). \tag{3.3}$$

More explicitly,

$$\begin{aligned}
G_1(\xi, \eta, \tau, \varepsilon, \nu) &= -\tau\varepsilon\eta - \frac{3}{16}(1-\nu)\xi'' + \frac{3}{16}(1-\nu)\xi \\
&\quad - (\xi - \varepsilon\eta)(\xi^2 + \varepsilon\eta^2) + (\xi^2 + \varepsilon\eta^2)^2\xi, \\
G_2(\xi, \eta, \tau, \varepsilon, \nu) &= \tau\xi - \frac{3}{16}(1-\nu)\eta'' + \frac{3}{16}(1-\nu)\eta \\
&\quad - (\xi + \eta)(\xi^2 + \varepsilon\eta^2) + (\xi^2 + \varepsilon\eta^2)^2\eta.
\end{aligned}$$

Clearly, (3.2) is equivalent to (3.1). We will solve the existence problem looking at the G formulation, and the stability problem looking at the F formulation.

We have already determined a solution of

$$G_1(\xi, \eta, \tau, 0, \nu) = 0.$$

it is given by $\xi = r$, and η and τ are arbitrary.

In next lemma, we state the sense in which we can find a solution for the second equation at $\varepsilon = 0$:

Lemma 3.1. *There exists a unique $\theta \in \mathbb{R}$ and a unique q in $H^2(\mathbb{R})$ and orthogonal to r which satisfy*

$$G_2(r, q, \theta, 0, \nu) = 0 \tag{3.4}$$

Moreover, q is even.

Proof. The second equation at $\varepsilon = 0$ can be written

$$\tau\xi - m\eta'' + m\eta - \xi^3 - \xi^2\eta + \xi^4\eta = 0.$$

Therefore, if we let $\xi = r$, we have to find q and θ such that (3.4) holds, i.e.

$$Bq = -mq'' + mq - r^2 q + r^4 q = r^3 - \theta r. \tag{3.5}$$

Define an unbounded operator B in $L^2(\mathbb{R})$ by

$$D(B) = H^2(\mathbb{R}), \quad Bu = -mu'' + Wu, \quad W = m - r^2 + r^4.$$

Clearly, B is self adjoint. By construction

$$Br = 0.$$

The lower bound of the essential spectrum of B is $m > 0$; since r is positive, the lower bound of the spectrum of B is zero, which is a simple isolated eigenvalue. Therefore, solving (3.5) will be possible if and only if its right hand side $r^3 - \theta r$ is orthogonal to $\ker B = \mathbb{R}r$. This defines θ thanks to

$$\theta = \frac{\int r^4\,dx}{\int r^2\,dx}.$$

Moreover, we define completely q by requiring

$$(q, r) = 0. \tag{3.6}$$

It is clear that the q we obtained belongs to $H^2(\mathbb{R})$. Since the equation is invariant by the transformation $x \mapsto -x$, q has to be even. $\square$

We will need precise asymptotic information, which we proceed to give now:

Lemma 3.2. *The number θ is a C^1 function of $\nu \in (0,1)$; we have the following asymptotics for θ and $\partial\theta/\partial L$:*

$$\theta = \frac{3}{4} - \frac{3}{8L} + O(\sqrt{\nu}). \tag{3.7}$$

$$\theta_1 = \frac{\partial \theta}{\partial L} = \frac{3}{8L^2} + O(L^{-1}\sqrt{\nu}). \tag{3.8}$$

Proof. Let us compute the difference between $3/4$ and θ:

$$\frac{3}{4} - \theta = \frac{\int_0^\infty (3r^2/4 - r^4)\,dx}{\int_0^\infty r^2\,dx}.$$

But an explicit computation and (2.10) give

$$\int_0^\infty R\,dx = \frac{3L}{4} + \frac{3}{8}\ln\frac{4}{3} + O(\sqrt{\nu}),$$

that is

$$\int_0^\infty R\,dx = \frac{3L}{4} + O(\sqrt{\nu}). \tag{3.9}$$

In the same fashion,

$$\begin{aligned}\int_0^\infty (3R/4 - R^2)\,dx &= \frac{3}{4}\int_0^\infty R(1 - 4R/3)\,dx \\ &= \frac{3}{4}\int_0^\infty \tilde{R}(1 - 4\tilde{R}/3)\,dx + O(\sqrt{\nu}) \\ &= \frac{3}{8}\tilde{R}(-L) + O(\sqrt{\nu}) = \frac{9}{32} + O(\sqrt{\nu}).\end{aligned}$$

Therefore,

$$\frac{3}{4} - \theta = \frac{3}{8L} + O(\sqrt{\nu}),$$

which implies immediately (3.7).

Since r is a C^∞ function of $\nu \in (0,1)$ with values in $H^2(\mathbb{R})$, and it is integrable as well as σ, θ is a C^1 function of ν. Let us compute $\partial\theta/\partial L$; we have

$$\frac{\partial \theta}{\partial L} = \left(\int_0^\infty r^2\,dx \int_0^\infty 4r^3\sigma\,dx - \int_0^\infty r^4\,dx \int_0^\infty 2r\sigma\,dx\right)\left(\int_0^\infty r^2\,dx\right)^{-2}. \tag{3.10}$$

But

$$\int_0^\infty 4r^3\sigma\,dx = -\int_0^\infty 4r^3 r'\,dx + O(\sqrt{\nu}) = R^2(0) + O(\sqrt{\nu}) = \frac{9}{16} + O(\sqrt{\nu}).$$

Similarly,

$$\int_0^\infty 2r\sigma\,dx = -\int_0^\infty 2rr'\,dx + O(\sqrt{\nu}) = R(0) + O(\sqrt{\nu}) = \frac{3}{4} + O(\sqrt{\nu}).$$

Therefore, the numerator of (3.10) is equal to

$$\begin{aligned}&\frac{9}{16}\int_0^\infty R\,dx - \frac{3}{4}\int_0^\infty R^2\,dx + O(L\sqrt{\nu}) = \frac{9}{16}\int_{-L}^\infty \tilde{R}(1 - 4\tilde{R}/3)\,dx + O(L\sqrt{\nu}) \\ &= \frac{27}{128} + O(L\sqrt{\nu}).\end{aligned}$$

With the help of (3.9), we obtain (3.8). ∎

Let us find now asymptotics for q and related quantities. Let

$$\phi = \frac{q}{r}. \tag{3.11}$$

Then ϕ satisfies the ordinary differential equation:

$$-m(\phi'' r + 2\phi' r') = r^3 - \theta r.$$

Multiplying by r we get

$$m(\phi' r^2)' = \theta r^2 - r^4. \tag{3.12}$$

Therefore,

$$m(\phi' r^2)(x) = a + \int_{-\infty}^{x} (\theta R - R^2)(y)\, dy.$$

But a must vanish: q' belongs to $L^2(\mathbb{R})$ and is equal to $\phi r' + \phi' r$; but

$$r' = -r \operatorname{sgn}(x) \sqrt{\Psi(r)/m},$$

hence

$$\phi r' = -q \operatorname{sgn}(x) \sqrt{\Psi(r)/m},$$

so that $\phi' r = q' - \phi r'$ belongs to $L^2(\mathbb{R})$. This implies that $\phi' r^2$ belongs to $L^2(\mathbb{R})$ and hence a vanishes. Therefore

$$m\phi'(x) = \frac{1}{R(x)} \int_{-\infty}^{x} (\theta - R) R\, dy = -\frac{1}{R(x)} \int_{x}^{\infty} (\theta - R) R\, dy. \tag{3.13}$$

Lemma 3.3. *The following asymptotics hold for $x \geq 0$:*

$$\phi'(x) = -2 - \frac{3\ln(1 - 4\tilde{R}(x-L)/3)}{4L\tilde{R}(x-L)} + O(\sqrt{\nu}/L). \tag{3.14}$$

$$\phi(0) = \frac{L}{3} + O(1), \tag{3.15}$$

$$\phi(L) = -\frac{2L}{3} + O(1) \tag{3.16}$$

$$|q|_{L^1(\mathbb{R})} = O(L^2), \quad |q|_{L^\infty(\mathbb{R})} = O(L), \quad |q| = O(L^{3/2}). \tag{3.17}$$

Proof. From (3.13), we obtain

$$m\phi'(x) = -\frac{1}{R(x)}(\theta - 3/4) \int_{x}^{\infty} R(y)\, dy - \frac{1}{R(x)} \int_{x}^{\infty} (3/4 - R) R\, dy. \tag{3.18}$$

For $x > 0$, we have thanks to (2.10)

$$\begin{aligned} \int_{x}^{\infty} (3/4 - R) R\, dy &= \int_{x-L}^{\infty} (3/4 - \tilde{R}) \tilde{R}\, dy + O(\sqrt{\nu} e^{-2x}) \\ &= \frac{3}{8} \tilde{R}(x - L) + O(\sqrt{\nu} e^{-2x}). \end{aligned}$$

But relation (2.10) implies also that for $x > 0$

$$\left| \frac{R(x)}{\tilde{R}(x-L)} - 1 \right| \leq O(1)\sqrt{\nu}\,. \tag{3.19}$$

Therefore,

$$\frac{1}{R(x)} \int_{-\infty}^{x} (3/4 - R) R\, dy = \frac{3}{8} + O(\sqrt{\nu}). \tag{3.20}$$

On the other hand,

$$\int_x^\infty R\,dy = \int_{x-L}^\infty \tilde R\,dy + \sqrt{\nu}\,e^{-2x}O(1)$$
$$= \frac{3}{8}\ln\bigl(1+4e^{2L-2x}/3\bigr) + \nu O(1)\frac{4\tilde R(x-L)}{1-4\tilde R(x-L)/3}.$$

Therefore, we have also

$$\frac{1}{R(x)}\int_x^\infty R\,dy = -\frac{3}{8}\frac{\ln\bigl(1-4\tilde R(x-L)/3\bigr)}{\tilde R(x-L)} + O\bigl(\sqrt{\nu}\,\bigr). \tag{3.21}$$

Thus we have obtained the asymptotic for $x \ge 0$

$$\int_x^\infty R(\theta - R)\,dy = R\left(\frac{3}{8} - \frac{3}{8}\left[\theta - \frac{3}{4}\right]\frac{\ln\bigl(1-4\tilde R(x-L)/3\bigr)}{\tilde R(x-L)} + O\bigl(\sqrt{\nu}\,\bigr)\right). \tag{3.22}$$

Relation (3.14) is an immediate consequence of (3.22).

In order to have an asymptotic for $\phi(0)$, we will give another asymptotic for ϕ', which will be much less precise, but more usable at this level of the discussion; nevertheless, the precise asymptotic (3.14) will be useful later.

For $x \ge L$,

$$\ln\bigl(1-4\tilde R(x-L)/3\bigr) = -\frac{4\tilde R(x-L)}{3} + \tilde R(x-L)^2 O(1).$$

Therefore, for $x \ge L$,

$$\phi'(x) = -2 + \frac{1}{L} + O(1)\frac{\tilde R(x-L)}{L} + O(\sqrt{\nu}).$$

In the same fashion, for $0 \le x \le L$,

$$-\ln\bigl(1-4\tilde R(x-L)/3\bigr) = \ln\bigl(1+e^{2(L-x)}\bigr) = 2(L-x) + O(1)e^{2(x-L)}$$

$$\phi'(x) = \begin{cases} -2x/L + O(1/L)e^{2(x-L)} & \text{if } 0 \le x \le L,\\ -2 + O(1/L) & \text{if } x \ge L.\end{cases} \tag{3.23}$$

By integration in x, we get now

$$\phi(x) - \phi(0) = \begin{cases} -x^2/L + O(1), & \text{if } 0 \le x \le L,\\ L - 2x + O(1) + (x-L)O(1/L), & \text{if } x \ge L.\end{cases} \tag{3.24}$$

The value of $\phi(0)$ is determined by the condition that q should be orthogonal to r (see Lemma 3.1). Therefore,

$$\phi(0)\int_0^\infty R(x)\,dx = \int_0^L \left(\frac{x^2}{L} + O(1)\right)R(x)\,dx$$
$$+ \int_L^\infty (-L+2x)R(x)\,dx + \int_L^\infty \left(O(1) + \frac{x-L}{L}O(1)\right)R(x)\,dx.$$

At this point we observe that

$$\int_0^\infty R\,dx = \frac{3L}{4} + O(1), \qquad \int_0^\infty \bigl(x^2/L + O(1)\bigr)R(x)\,dx = \frac{L^2}{4} + O(L),$$
$$\int_L^\infty (2x-L)R(x)\,dx = O(L), \qquad \int_L^\infty \bigl(O(1) + O(1)(x-L)/L\bigr)R(x)\,dx = O(1).$$

This yields the first relation in (3.15). The second relation in (3.16) is a consequence of the first together with (3.24).

Estimates (3.17) are a direct consequence of the definition (3.11) of ϕ, of the equivalents (3.23) and of (3.15) and (3.16). $\square$

Since 0 is an isolated eigenvalue of B for all ν in $(0,1)$ and since $(1-\nu)^{-1}W$ is an infinitely differentiable function of ν with values in the space C_b^2 of functions bounded as well as their first two derivatives in space, we can see that q is a C^∞ mapping from $\nu \in (0,1)$ to $H^2(\mathbb{R})$, under the normalizing condition (3.6).

Differentiation of q with respect to L will be useful too; define

$$q_1 = \frac{\partial}{\partial L} q(x, 4e^{-4L}).$$

If we differentiate (3.5) with respect to L we obtain

$$Bq_1 + (4r^3 - 2r)q\sigma = \hat{\rho} + 3r^2\sigma - \theta\sigma - \theta_1 r, \tag{3.25}$$

where

$$\hat{\rho} = \frac{4\nu}{1-\nu}(r^4 q - r^2 q - r^3 + \theta r). \tag{3.26}$$

We recall that θ_1 has been defined at (3.8)

We will need an estimate over ϕ''. From (3.12), we have

$$m\phi'' = -R + \theta - 2m\phi' r'/r.$$

Assume that $x \leq 0$ and Write

$$a = \frac{\phi'(x) - 2}{2};$$

then, thanks to (2.17), we have

$$-2m\phi' r'/r = -2(1+a)mR' = -(1-\nu)(1+a)\left(\frac{3}{4} - R + O(\sqrt{\nu}\right),$$

so that

$$\theta - R - -2m\phi' r'/r = \theta - \frac{3}{4} - \left(\frac{3}{4} - R\right) a + O(\sqrt{\nu}).$$

But the following expressions are bounded independently of ν for $x \leq 0$:

$$\left(\frac{3}{4} - \tilde{R}(|x| - L)\right) \frac{\ln(1 - 4\tilde{R}(|x| - L)/3)}{\tilde{R}(|x| - L)} \text{ and } \sqrt{\nu} e^x \frac{\ln(1 - 4\tilde{R}(|x| - L)/3)}{\tilde{R}(|x| - L)};$$

therefore, the term $a(3/4 - R)$ is an $O(1/L)$, and we conclude with the help of (3.7) that

$$|\phi''|_\infty = O(L^{-1}). \tag{3.27}$$

We will give now two estimates on the second lowest eigenvalue μ_2 of B; this eigenvalue is the lowest eigenvalue of B restricted to odd functions.

Lemma 3.4. *The second lowest eigenvalue μ_2 of B satisfies*

$$\mu_2 = \frac{m\pi^2}{4L^2} + O(L^{-5/2}). \tag{3.28}$$

Proof. If t_2 is an eigenvalue of B, it is odd, thanks to the general Sturm-Liouville theory. Define an operator B_0 in $L^2(\mathbb{R}^+)$ by

$$D(B_0) = H_0^1(\mathbb{R}^+) \cap H^2(\mathbb{R}^+), \quad B_0 u = Bu\mid_{\mathbb{R}^+};$$

then μ_2 is the first eigenvalue of B_0. In particular, for all $v \in H_0^1(\mathbb{R}^+) \setminus \{0\}$, the following inequality holds:

$$\mu_2 \leq \int_0^\infty \left(m\left|v'\right|^2 + Wv^2\right)\left(\int_0^\infty v^2\,dx\right)^{-1}.$$

Define a test function v by

$$v(x) = \begin{cases} \sin\bigl(\pi x/2\bigl(L-\sqrt{L}\bigr)\bigr) & \text{if } 0 \le x \le L-\sqrt{L}, \\ r(x)/r\bigl(L-\sqrt{L}\bigr). & \text{if } x \ge L-\sqrt{L}. \end{cases}$$

For all large enough L, $W \le 0$ on $\bigl[0, L-\sqrt{L}\bigr]$; therefore an integration by parts on $\bigl[L-\sqrt{L}, +\infty\bigr)$ shows that

$$\int_0^\infty \bigl(m\,|v'|^2 + Wv^2\bigr)\,dx \le \frac{\pi^2 m}{8\bigl(L-\sqrt{L}\bigr)^3} - \frac{R'\bigl(L-\sqrt{L}\bigr)}{R(L-\sqrt{L}}.$$

On the other hand,

$$\int_0^{L-\sqrt{L}} v^2\,dx \ge \frac{L-\sqrt{L}}{2};$$

therefore, we see now that

$$\mu_2 \le \pi^2 m/8L^2 + O(L^{-5/2}. \tag{3.29}$$

Conversely, define $L_1 > 0$ by

$$R(L_1) = \frac{1}{2}\left(1 + \frac{\sqrt{1+3\nu+16\mu_2}}{2}\right);$$

then, thanks to (2.10) and (3.29)

$$L_1 = L + \frac{1}{2}\ln 2 + O(1/L^2);$$

the function $x \mapsto \mu_2 - W(x)$ changes sign only at $x = L_1$; therefore, there exists $L_2 \le L_1$ such that t_2 increases on $[0, L_2]$ and decreases on $[L_2, \infty)$.

Case 1: $L_2 \le L - \sqrt{L}$**.** Write $M = \sqrt{\mu_2/m}$; since t_2 and t_2' cannot vanish simultaneously and $t_2(0)$ vanishes, there exists a continuous determination of the angle γ such that

$$t_2 = b\sin M\gamma, \quad t_2' = bM\cos M\gamma, \quad \gamma(0) = 0. \tag{3.30}$$

A classical calculation gives

$$1 - \gamma' = \frac{W}{mM^2}\sin^2 M\gamma.$$

There exists a constant C such that for all L large enough,

$$\forall x \in \bigl[0, L-\sqrt{L}\bigr], \quad 0 \le -W(x) \le Ce^{-2\sqrt{L}};$$

if we let $\varepsilon^2 = Cme^{-2\sqrt{L}}$, γ satisfies the differential inequality by integration of the inequality

$$\gamma' \le 1 + \varepsilon\gamma(x)^2,$$

which we integrate as:

$$\gamma(x) \le \varepsilon^{-1}\tan\bigl(\varepsilon x\bigr). \tag{3.31}$$

Thanks to (3.30), $M\gamma(L_2) = \pi/2$; therefore thanks to (3.31),

$$M \ge \varepsilon\pi/2\tan(\varepsilon L_2);$$

but the definition of ε and the condition on L_2 imply immediately that

$$M \ge \frac{\pi}{2(1+C\varepsilon^2)\bigl(L-\sqrt{L}\bigr)}$$

which implies immediately the desired estimate (3.28).

Case 2: $L_2 \geq L - \sqrt{L}$. We have to give a different argument: multiply the equation $-mt_2'' + Wt_2 = \mu_2 t_2$ by v and perform two integration by parts on the term involving a second derivative; we eventually obtain

$$\left(\frac{m\pi^2}{4\big(L-\sqrt{L}\big)^2} - \mu_2\right)\int_0^{L-\sqrt{L}} t_2 v\,dx = -\int_0^{L-\sqrt{L}} t_2 W v\,dx + \mu_2 \int_{L-\sqrt{L}}^{\infty} t_2 v\,dx$$
$$+ mt_2(L-\sqrt{L})\big[v'\big(L-\sqrt{L}-0\big) - v'\big(L-\sqrt{L}+0\big)\big].$$

Normalize t_2 so that $t_2(L_2) = 1$; then, by concavity on $[0, L_2]$,

$$t_2(x) \geq x/L_2 \geq x/L_1;$$

therefore, there exists $C > 0$ such that

$$\int_0^{L-\sqrt{L}} t_2 v\,dx \geq CL;$$

on the other hand,

$$\left|\int_0^{L-\sqrt{L}} t_2 W v\,dx\right| = O\big(Le^{-2\sqrt{L}}\big), \quad \int_{L-\sqrt{L}}^{\infty} t_2 v\,dx = O\big(\sqrt{L}\big),$$
$$\left|v'\big(L-\sqrt{L}-0\big) - v'\big(L-\sqrt{L}+0\big)\right| = O\big(e^{-2\sqrt{L}}\big)$$

Therefore

$$CL\left|\frac{m\pi^2}{4\big(L-\sqrt{L}\big)^2} - \mu_2\right| = O\big(L^{-3/2}\big),$$

which implies immediately (3.29).

This proves Lemma 3.4. □

4. Existence of pulses

We will prove that there are couples (ξ, η) which satisfy (3.2); the main idea is to use an ansatz suggested by previous long computations; this ansatz relies on the observation that for small values of ε, the pulse is simply stretched; i.e., the new pulse obtained should be very close to a pulse with $\varepsilon = 0$ but a smaller value of ν. In fact, we will work in reverse: given ν, we define $\nu^\flat = \nu e^{-4y}$, with $y \geq 0$; we let $L^\flat = L + y$, $r^\flat = r(\cdot, \nu^\flat)$, $q^\flat = q(\cdot, \nu^\flat)$, $\theta^\flat = \theta(\cdot, \nu^\flat)$, $s^\flat = s(\cdot, \nu^\flat)$ and we have to determine an ansatz $\varepsilon^\flat$ for ε. We will start with this ansatz, with very little motivation. The justification will be provided by the proofs which come later.

Once the ansatz is determined, we get an asymptotic description of the behavior of $\varepsilon^\flat$. Then, we go on to apply a version of the implicit function theorem *cum* estimates which is given in the appendix; for this purpose, we have to check that $G(r^\flat, q^\flat, \theta^\flat, \varepsilon^\flat, \nu)$ is very small, that $D_{(\xi,\eta,\tau,\varepsilon)}G(r^\flat, q^\flat, \theta^\flat, \varepsilon^\flat, \nu)$ is invertible, with an estimate on the norm of its inverse; this necessitates an adequate choice of functional spaces; finally, we check that the second derivatives of G are not too large in a neighborhood of $(r^\flat, q^\flat, \theta^\flat, \varepsilon^\flat)$, and we are able to apply this implicit function theorem.

Moreover, we obtain precise asymptotics on the second term of the expansion. These estimates are crucial for the proof of the skew stabilization.

Define

$$g_1(\varepsilon) = G_1(r^\flat, q^\flat, \theta^\flat, \varepsilon, \nu). \tag{4.1}$$

Another expression of g_1 obtained by substituting the value of $(-r''+r)^\flat$ from (2.3) is

$$g_1(\varepsilon) = \frac{\nu - \nu^\flat}{1-\nu^\flat}(r^5 - r^3)^\flat + \varepsilon(r^2 q + 2r^3 q^2 - rq^2 - \theta q)^\flat + \varepsilon^2 (rq^4 + q^3)^\flat. \tag{4.2}$$

The ansatz for ε is given by

$$(g_1(\varepsilon^\flat), s^\flat) = 0. \tag{4.3}$$

Our purpose now is to show that this ansatz determines uniquely $\varepsilon^\flat$ for small enough values of ν, and for all $y \geq 0$. Let us introduce new notations

$$\kappa = \frac{\nu - \nu^\flat}{1-\nu^\flat}, \tag{4.4}$$

$$\begin{aligned} a_0 &= (r^5 - r^3, s), \\ a_1 &= (r^2 q + 2r^3 q^2 - rq^2 - \theta q, s), \\ a_2 &= (rq^4 + q^3, s). \end{aligned} \tag{4.5}$$

Of course, $a_j^\flat$ will be equal to a_j with r, q, θ and s replaced by their analogues $r^\flat$, $q^\flat$, $\theta^\flat$ and $s^\flat$.

Therefore, (4.3) is equivalent to the following equation of the second degree:

$$\kappa a_0^\flat + \varepsilon^\flat a_1^\flat + (\varepsilon^\flat)^2 a_2^\flat = 0. \tag{4.6}$$

In order to have an asymptotic for $\varepsilon^\flat$ we will need an asymptotic for a_j, $0 \leq j \leq 2$.

We introduce the notation

$$\mathcal{O}(1)$$

for any quantity whose absolute value or relevant norm is bounded by some power of L.

Lemma 4.1. *The following asymptotics hold*

$$a_0 = -\frac{9}{64} + O(\sqrt{\nu}), \tag{4.7}$$

$$a_1 = \frac{9}{16} + O\left(\frac{1}{L}\right), \tag{4.8}$$

$$a_2 = \frac{L^4}{36} + O(L^3). \tag{4.9}$$

Proof. We have

$$\begin{aligned} (r^5 - r^3, s) = (r^5 - r^3, \sigma) + O(\nu L) &= -2\int_0^\infty (r^5 - r^3) r'\, dx + O(\sqrt{\nu}) \\ &= -\frac{9}{64} + O(\sqrt{\nu}). \end{aligned}$$

This gives (4.7).

The second object is more difficult. If we define

$$a = (r^2 q + 2r^3 q^2 - rq^2 - \theta q, \sigma) \tag{4.10}$$

estimate (2.40) implies that $|a_1 - a| = \nu\mathcal{O}(1)$. But, thanks to (3.5) and (3.25), $\tilde{a}$ can be rewritten as

$$a = \big((r^2 - \theta)q, \sigma\big) + \frac{1}{2}(\hat{\rho}, q) + \frac{1}{2}(3r^2\sigma - \theta\sigma - \theta_1 r, q) - \frac{1}{2}(q_1, r^3 - Br).$$

Thus, we have

$$a = -(r^4 - \theta r^2, \phi_1) + \frac{1}{2}\frac{\partial}{\partial L}(r^4 - \theta r^2, \phi) + \frac{1}{2}(\hat{\rho}, q).$$

We transform this expression using (3.12):

$$a = \big(m(r^2\phi')', \phi_1\big) - \frac{1}{2}\frac{\partial}{\partial L}\big(m(r^2\phi')', \phi\big) + +\frac{1}{2}(\hat{\rho}, q), \tag{4.11}$$

and by integration by parts

$$a = m(r\sigma\phi', \phi') + \frac{1}{2}\frac{\partial m}{\partial L}\big(r\phi', r\phi'\big) + \frac{1}{2}\big(\hat{\rho}, q\big). \tag{4.12}$$

But

$$\begin{aligned}\int r\sigma\left|\phi'\right|^2\,dx &= -2\int_0^\infty rr'\left|\phi'\right|^2\,dx + O(\sqrt{\nu})\\ &= -R\left|\phi'\right|^2\Big|_0^\infty + 2\int_0^\infty R\phi'\phi''\,dx + O(\sqrt{\nu})\\ &= \frac{3}{4}\left|\phi'(L)\right|^2 + 2\int_0^L\left(R - \frac{3}{4}\right)\phi'\phi''\,dx + 2\int_L^\infty R\phi'\phi''\,dx.\end{aligned}$$

From estimate (3.27) over ϕ'', and the following facts

$$\int_0^L |R - 3/4|\,dx = O(1), \qquad \int_L^\infty R\,dx = O(1),$$

we can see that

$$\int r\sigma\left|\phi'\right|^2\,dx = 3\left|\phi'(L)\right|^2/4 + O(1/L), \tag{4.13}$$

and from asymptotic (3.23), we obtain

$$\int r\sigma\left|\phi'\right|^2\,dx = 3 + O(1/L).$$

This implies (4.8).

The last of the three estimates concerns

$$\int (rq^4 + q^3)s\,dx.$$

Thanks to (3.17) and (2.35), the second term will contribute an $O(L^3)$. Let us estimate $\int rq^4 s\,dx$, which will be equivalent to CL^4. More precisely,

$$\begin{aligned}\int rq^4 s\,dx &= \int rq^4\sigma\,dx + \nu\mathcal{O}(1)\\ &= -2\int_0^\infty rr'r^4\phi^4\,dx + \nu\mathcal{O}(1)\\ &= -\frac{R^3}{3}\phi^4\Big|_0^\infty + \int_0^\infty \frac{R^3}{3}(\phi^4)'\,dx + \nu\mathcal{O}(1)\\ &= \frac{R^3(0)}{3}\phi^4(L) + \int_0^L \frac{R^3 - R^3(0)}{3}(\phi^4)'\,dx + \int_L^\infty \frac{R^3}{3}(\phi^4)'\,dx + \nu\mathcal{O}(1).\end{aligned}$$

From the asymptotics (3.14) on ϕ' and (3.15) and (3.16) on ϕ, we can see that

$$\int_0^L \left|\big(R^3 - R^3(0)\big)(\phi^4)')\right|\,dx = O(L^3)\int_0^L \left|R^3 - R^3(0)\right|\,dx = O(L^3)$$

and

$$\int_L^\infty R^3\left|(\phi^4)'\right|\,dx \le C\int_L^\infty x^3 e^{-2(x-L)}\,dx = O(L^3).$$

We obtain now

$$\int rq^4 s\,dx = \frac{1}{3}\left(\frac{3}{4}\right)^3\left(-\frac{2L}{3}\right)^4 + O(L^3) = \frac{L^4}{36} + O(L^3),$$

which is precisely the last of the three estimate of our list. □

Therefore, for all ν small enough, and for all $y \geq 0$, the discriminant of equation (4.6) is strictly positive, and there exists a unique positive solution of this equation, which is given by

$$\varepsilon^\flat = \frac{-2a_0^\flat \kappa}{a_1^\flat + \sqrt{(a_1^\flat)^2 - 4\kappa a_0^\flat a_2^\flat}}. \tag{4.14}$$

From this relation we get

$$\varepsilon^\flat = -\frac{a_0^\flat \kappa}{a_1^\flat} + \kappa^2 \mathcal{O}(1) \text{ and } \frac{\varepsilon^\flat}{\kappa} = \frac{1}{4} + O\left(\frac{1}{L}\right). \tag{4.15}$$

We will apply now lemma 7.1. Define

$$U = (\xi, \eta, \tau, \varepsilon), \quad U^\flat = (r^\flat, q^\flat, \theta^\flat, \varepsilon^\flat). \tag{4.16}$$

Thus, we have to check that $G(U^\flat, \nu)$ is small, and to obtain bounds for the inverse of $D_U G(U^\flat, \nu)$ and for the second derivative of G in a neighborhood of $U^\flat$.

The first result is a bound on $G(U^\flat, \nu)$.

Lemma 4.2. *Assume that $y \in [0, L^p]$, where $p \geq 1$ is an integer. The following bound holds:*

$$|G(U^\flat, \nu)| = \kappa \mathcal{O}(1).$$

Proof. We have already obtained an expression for $G_1(U^\flat, \nu)$: it is given by

$$G_1(U^\flat, \nu) = \kappa(r^5 - r^3)^\flat + \varepsilon^\flat (r^2 q + 2r^3 q^2 - rq^2 - \theta q)^\flat + (\varepsilon^\flat)^2 (rq^4 + q^3)^\flat.$$

We can see that

$$|(r^5 - r^3)^\flat| + |(r^2 q + 2r^3 q^2 - rq^2 - \theta q)^\flat| + |(rq^4 + q^3)^\flat| = \mathcal{O}(1).$$

With the help of (4.15), we obtain

$$|G_1(U^\flat, \nu)| = \kappa \mathcal{O}(1). \tag{4.17}$$

An expression for $G_2(U^\flat, \nu)$ is given by

$$G_2(r^\flat, q^\flat, \theta^\flat, \varepsilon^\flat, \nu) = \kappa(r^4 q - r^3 - r^2 q + r\theta)^\flat + \varepsilon^\flat (2r^2 q^3 - rq^2 - q^3)^\flat + (\varepsilon^2 q^5)^\flat.$$

Arguing as above,

$$|G_2(U^\flat, \nu)| = \kappa \mathcal{O}(1). \tag{4.18}$$

Relations (4.17) and (4.18) imply the assertion of the lemma. □

Let us compute now the differential of G with respect to U, at $(U^\flat, \nu)$:

$$\begin{aligned}
D_\xi G_1(U^\flat, \nu) &= -m\frac{d^2}{dx^2} + m + (5r^4 - 3r^2)^\flat + \left[\varepsilon(2rq - q^2 + 6r^2 q^2) + \varepsilon^2 q^4\right]^\flat \\
&= \frac{1-\nu}{1-\nu^\flat} A^\flat + \kappa(5r^4 - 3r^2)^\flat + \left[\varepsilon(2rq - q^2 + 6r^2 q^2) + \varepsilon^2 q^4\right]^\flat, \\
D_\eta G_1(U^\flat, \nu) &= \left[\varepsilon(r^2 - 2rq - \theta + 4r^3 q) + \varepsilon^2(3q^2 + 4rq^3)\right]^\flat, \\
D_\tau G_1(U^\flat, \nu) &= -\left[\varepsilon q\right]^\flat, \\
D_\varepsilon G_1(U^\flat, \nu) &= \left[-\theta q + r^2 q - rq^2 + 2r^3 q^2 + \varepsilon(2q^3 + 2rq^4)\right]^\flat,
\end{aligned}$$

and, similarly,

$$D_\xi G_2(U^\flat,\nu) = \left[\theta - 3r^2 - 2rq + 4r^3q + \varepsilon(-q^2+4rq^3)\right]^\flat,$$
$$D_\eta G_2(U^\flat,\nu) = \frac{1-\nu}{1-\nu^\flat}B^\flat + \kappa(r^4-r^2)^\flat + \left[\varepsilon(6r^2q^2-3q^2-2rq)+5\varepsilon^2q^4\right]^\flat,$$
$$D_\tau G_2(U^\flat,\nu) = r^\flat,$$
$$D_\varepsilon G_2(U^\flat,\nu) = \left[-rq^2 - q^3 + 2r^2q^3 + 2\varepsilon q^5\right]^\flat.$$

Therefore, we may write

$$D_U G(U^\flat,\nu) = (1-\kappa)\mathcal{G}^\flat + \kappa\mathcal{H},$$

where

$$\mathcal{G} = \begin{pmatrix} A & 0 & 0 & z \\ w & B & r & \hat{z} \end{pmatrix}$$

and the letters w, z and $\hat{z}$ denote multiplication operators by the functions

$$w = \theta - 3r^2 - 2rq + 4r^3q,$$
$$z = -\theta q + r^2q - rq^2 + 2r^3q^2$$
$$\hat{z} = -rq^2 - q^3 + 2r^2q^3.$$

The matrix $\mathcal{H}$ has components H_{ij}, $1 \le i \le 2, 1 \le j, \le 4$ which satisfy

$$\max_{i,j} |H_{ij}|_\infty = \mathcal{O}(1).$$

Lemma 4.3. *Let E_2 be the space*

$$E_2 = \{\xi \in H^2_{even}(\mathbb{R}), \xi \perp s\} \times \{\eta \in H^2_{even}(\mathbb{R}), \eta \perp r\} \times \mathbb{R}^2$$

and let E_0 be the space

$$E_0 = (L^2_{even}(\mathbb{R}))^2.$$

The space $E_2^\flat$ is defined in an obvious way. Then, for all $p < \infty$, for all small enough ν and for $y \le L^p$, $D_U G(U^\flat,\nu)$ is an isomorphism from $E_2^\flat$ to E_0, and the norm of its inverse is bounded by $\mathcal{O}(1)$.

Proof. We first study $\mathcal{G}$ as an operator from E_2 to E_0. Given $h = (h_1,h_2)^T \in E_0$, we first solve

$$\mathcal{G}U_1 = h, \tag{4.19}$$

where $U_1 = (\xi_1,\eta_1,\tau_1,\varepsilon_1)^T$ is to be found in E_2. Equation (4.18) is equivalent to the system

$$\begin{cases} A\xi_1 + z\varepsilon_1 = h_1, \\ w\xi_1 + B\eta_1 + r\tau_1 + \hat{z}\varepsilon_1 = h_2. \end{cases} \tag{4.20}$$

Since ξ_1 is perpendicular to s, the first equation gives

$$(z,s)\varepsilon_1 = (h_1,s). \tag{4.21}$$

But (z,s) is precisely equal to a_1 defined at (4.5). Therefore, there exists a unique ε_1 satisfying (4.21) and

$$|\varepsilon_1| \le \mathcal{O}(1)|h|.$$

Restricted to even functions perpendicular to s, A is invertible, and its inverse is bounded thanks to Theorem 2.7. Therefore

$$|\xi_1| = \mathcal{O}(1),$$

and by classical elliptic estimates

$$|\xi_1|_{H^2(\mathbb{R})^2} = \mathcal{O}(1).$$

For the second equation of (4.19) to have a solution, the following orthogonality condition must be satisfied:

$$h_2 - \hat{z}\varepsilon_1 - w\xi_1 - r\tau_1 \perp r.$$

Hence

$$\tau_1 = \frac{(h_2 - \hat{z}\varepsilon_1 - w\xi_1, r)}{|r|^2},$$

which implies immediately that

$$|\tau_1| \leq \|h\|_{E_0}\mathcal{O}(1).$$

The operator B restricted to $r^\perp$ is invertible, and the norm of its inverse is estimated by $\mu_2^{-1} = O(L^2)$, thanks to Lemma 3.4. Therefore, the unique η_1 orthogonal to r which solves the second equation of (4.19) satisfies

$$|\eta_1| \leq \|h\|_{E_0}\mathcal{O}(1).$$

By classical elliptic estimates

$$|\eta_1|_{H^2(\mathbb{R})} \leq \|h\|_{E_0}\mathcal{O}(1),$$

and therefore

$$\|U_1\|_{E_2} \leq \|h\|_{E_0}\mathcal{O}(1).$$

The full system

$$(1-\kappa)\mathcal{G}^\flat U_1 + \kappa\mathcal{H}U_1 = h$$

can be rewritten

$$U_1 + \frac{\kappa}{1-\kappa}(\mathcal{G}^\flat)^{-1}\mathcal{H}U_1 = (\mathcal{G}^\flat)^{-1}\frac{h}{1-\kappa}.$$

Clearly,

$$\|(\mathcal{G}^\flat)^{-1}\mathcal{H}\|_{\mathcal{L}(E_2^\flat)} = \mathcal{O}(1).$$

Therefore, for ν small enough and $y \leq L^p$, we can see that

$$\left\|\frac{\kappa}{1-\kappa}(\mathcal{G}^\flat)^{-1}\mathcal{H}\right\|_{\mathcal{L}}(E_2^\flat) \leq \frac{1}{2},$$

and our assertion is proved. □

We have to estimate $D^2G(U,\varepsilon,\nu)$ in a neighborhood of $(U^\flat,\nu)^T$. Let us choose for this neighborhood the ball of radius 1 around $U^\flat$, in the space $E_2^\flat$. Without any calculation, we can see that D^2G contains only multiplications by functions which are polynomials in ξ, η, τ, ε and ν. Therefore, the norm of this operator is $\mathcal{O}(1)$ for $y \leq L^p$.

Now, these results enable us to apply Lemma 7.1. There exists for all small enough ν a unique $U \in E_2$ such that

$$G(U,\nu) = 0.$$

Moreover,

$$\begin{cases} \|U - U^\flat\|_{E_2^\flat} = \kappa\mathcal{O}(1), \\ \|U - U^\flat + D_UG(U^\flat,\nu)^{-1}G(U^\flat,\nu)\|_{E_2^\flat} = \kappa^2\mathcal{O}(1). \end{cases} \tag{4.22}$$

It will be useful in what follows to have an asymptotic for

$$U_1 = D_UG(U^\flat,\nu)^{-1}G(U^\flat,\nu).$$

The information needed is summarized in Lemma 4.4. But we need to introduce new notations. The orthogonal projections P on s, $P_\perp$ on $s^\perp$, Q on r and $Q_\perp$ on $r^\perp$ are defined by

$$\begin{cases} Pv = \dfrac{(s,v)s}{|s|^2}, \quad Qv = \dfrac{(r,v)r}{|r|^2}, \\ P_\perp v = v - Pv, \quad Q_\perp v = v - Qv \end{cases}$$

We will also abuse notations: when f is orthogonal to r' we denote by $A^{-1}f$ the unique solution u of $Au = f$ which is orthogonal to r'; the analogous convention will be used for B.

Lemma 4.4. *Let*

$$U_1 = D_U G(U^\flat, \nu)^{-1} G(U^\flat, \nu) = (\xi_1, \eta_1, \tau_1, \varepsilon_1)^T.$$

The following asymptotics hold

$$\xi_1 = (A^\flat)^{-1} G_1(U^\flat, \nu) + \kappa^2 \mathcal{O}(1) \tag{4.23}$$

$$\eta_1 = (B^\flat)^{-1} Q_\perp^\flat \big(G_2(U^\flat, \nu) - w^\flat \xi_1\big) + \kappa^2 \mathcal{O}(1), \tag{4.24}$$

$$\tau_1 = \frac{\big(G_2(U^\flat, \nu) - \xi_1 w^\flat, r^\flat\big)}{|r^\flat|^2} + \kappa^2 \mathcal{O}(1), \tag{4.25}$$

$$\varepsilon_1 = \kappa^2 \mathcal{O}(1). \tag{4.26}$$

Proof. The vector U_1 satisfies the equation

$$\mathcal{G}^\flat U_1 + \frac{\kappa}{1-\kappa} \mathcal{H} U_1 = \frac{G(U^\flat, \nu)}{1-\kappa}. \tag{4.27}$$

The first equation of (4.27) yields

$$(z^\flat, s^\flat)\varepsilon_1 + \frac{\kappa}{1-\kappa}\big((\mathcal{H}U_1)_1, s^\flat\big) = 0.$$

Since $\|U_1\|_{E_2^\flat} = \kappa \mathcal{O}(1)$, we can see that (4.26) holds. Therefore,

$$A^\flat \xi_1 = G_1(U^\flat, \nu) + \kappa^2 \mathcal{O}(1),$$

which together with the orthogonality $\xi_1 \perp s^\flat$ implies (4.23). In a similar fashion,

$$\xi_1 w^\flat + B^\flat \eta_1 + r^\flat \tau_1 + \hat{z}^\flat \varepsilon_1 = \frac{G_2(U^\flat, \nu)}{1-\kappa} - \frac{\kappa}{1-\kappa} (\mathcal{H}U_1)_2,$$

so that

$$|r^b|^2 \tau_1 = \left(\frac{G_2(U^\flat, \nu)}{1-\kappa} - w^\flat \xi_1, r^\flat \right) + \kappa^2 \mathcal{O}(1).$$

Hence (4.25) holds, and (4.24) is inferred immediately from (4.25). □

The only remaining task is to check that the (ξ, η) obtained in this fashion yields an $u = \xi + i\sqrt{\varepsilon}\,\eta$ which has the adequate asymptotic behavior at $x = \infty$. The linearized equation at infinity for u is

$$-\frac{3}{16}(1-\nu)u'' + \frac{3}{16}(1-\nu)u + i\sqrt{\varepsilon}\tau u = 0.$$

Therefore, if β is the square root of $\big(1 + i\sqrt{\varepsilon}\,\tau\big)/m$ which has positive real part, there is a constant γ such that

$$u(x) \sim \gamma e^{-i\beta|x|}.$$

The phase of u is given by

$$\arg u(x) = \arctan \frac{\sqrt{\varepsilon}\eta(x)}{\xi(x)}.$$

According to our ansatz, this is very close to

$$\arctan\bigl(\sqrt{\varepsilon}\,\phi^\flat(x)\bigr),$$

whose behavior is exactly the behavior we postulated at the beginning of this article.

Finally, we obtain:

Theorem 4.5. *For all $p > 0$, for all ν small enough, there exists $C > 0$ such that for all α satisfying*

$$|\alpha| \le \alpha_m = \frac{1}{2}\sqrt{\nu}\,\bigl(1 - CL^{-p}\bigr), \tag{4.28}$$

there exists a pulse solution of (1.7)*, i.e. a solution of the form*

$$u(x,t) = e^{i\omega t} r(x) e^{i\phi(x)},$$

where r is a positive function which decays exponentially at infinity, and $\phi(x)$ is asymptotic at infinity to $-C|x| + D$, with C a positive constant and D a real constant.

Proof. We just have to translate the condition on α from the condition on $\varepsilon^\flat$. We know from (4.15) that there exist C', k and K such that

$$\varepsilon^\flat \ge \left(\frac{1}{4} - \frac{C'}{L+y}\right)\kappa - \nu^2 K L^k$$

Since the function

$$y \mapsto \left(\frac{1}{4} - \frac{C'}{L+y}\right)\frac{1 - e^{-4y}}{1 - 4\nu e^{-4y}}$$

is increasing on $\mathbb{R}^+$ for ν small enough, we can see that if

$$\alpha^2 \le \left(\frac{1}{4} - \frac{C'}{L+L^p}\right)\nu\frac{1 - e^{-4L^p}}{1 - 4\nu e^{-4L^p}} - \nu^2 K L^k, \tag{4.29}$$

the existence theory works; but it is clear that for ν small enough, the right hand side of (4.29) is at least equal to $\nu(1 - C/L^p)^2/4$, and the theorem is proved. ☐

5. Stability of the pulse

Our purpose is to study now the spectrum of

$$D_u F(\xi + i\sqrt{\varepsilon}\,\eta, \sqrt{\varepsilon}\,\tau, \sqrt{\varepsilon}, \nu),$$

under the assumption

the parameter ν is small enough and there exists $p \ge 1$ such that $0 \le y \le L^p$.

We define

$$\mathcal{D} = D_u F(\xi + i\sqrt{\varepsilon}\,\eta, \sqrt{\varepsilon}\,\tau, \sqrt{\varepsilon}, \nu)/(1-\kappa), \tag{5.1}$$

and we observe that

$$\mathcal{D} = \mathcal{A}^\flat + \sqrt{\kappa}\mathcal{B} + \kappa\mathcal{C}, \tag{5.2}$$

where

$$\mathcal{A}^\flat = \begin{pmatrix} A^\flat & 0 \\ 0 & B^\flat \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 0 & \mathcal{B}_{12} \\ \mathcal{B}_{21} & 0 \end{pmatrix} \text{ and } \mathcal{C} = \begin{pmatrix} \mathcal{C}_{11} & 0 \\ 0 & \mathcal{C}_{22} \end{pmatrix}. \tag{5.3}$$

More precisely, we have

$$\mathcal{B}_{12} = \frac{\sqrt{\varepsilon}(-\tau + \xi^2 - 2\xi\eta + 4\xi^3\eta) + \varepsilon^{3/2}(3\eta^2 + 4\xi\eta^3)}{\sqrt{\kappa}\,(1-\kappa)} \tag{5.4}$$

$$\mathcal{B}_{21} = \frac{\sqrt{\varepsilon}(\tau - 3\xi^2 - 2\xi\eta + 4\xi^3\eta) + \varepsilon^{3/2}(-\eta^2 + 4\xi\eta^3)}{\sqrt{\kappa}\,(1-\kappa)} \tag{5.5}$$

$$\mathcal{C}_{11} = \left[\frac{5\xi^4 - 3\xi^2 + \varepsilon(2\xi\eta - \eta^2 + 6\xi^2\eta^2) + \varepsilon^2\eta^4}{1-\kappa} - (5r^4 - 3r^2)^\flat\right]\frac{1}{\kappa} \tag{5.6}$$

$$\mathcal{C}_{22} = \left[\frac{\xi^4 - \xi^2 + \varepsilon(-2\xi\eta - 3\eta^2 + 6\xi^2\eta^2) + 5\varepsilon^2\eta^4}{1-\kappa} - (r^4 - r^2)^\flat\right]\frac{1}{\kappa}. \tag{5.7}$$

It is immediate from (4.22), (4.23), (4.24), (4.25) and (4.26) that the following estimate holds:

$$\|\mathcal{B}_{12}\|_{L^\infty(\mathbb{R})} + \|\mathcal{B}_{21}\|_{L^\infty(\mathbb{R})} + \|\mathcal{C}_{11}\|_{L^\infty(\mathbb{R})} + \|\mathcal{C}_{22}\|_{L^\infty(\mathbb{R})} = \kappa\mathcal{O}(1).$$

Identifying the multiplication by $\mathcal{B}$ or $\mathcal{C}$ to an operator from $L^2(\mathbb{R})^2$ to itself, we see that

$$\|\mathcal{B}\|_{\mathcal{L}} + \|\mathcal{C}\|_{\mathcal{L}} = \mathcal{O}(1). \tag{5.8}$$

The first result concerns the dimension of the generalized eigenspace associated to the eigenvalues of $\mathcal{D}$ contained inside a conveniently small circle; for this purpose, we let $K_1 > 0$ be a constant such that for all L large enough

$$\mu_2 + \lambda \geq 4/\big(K_1L^2\big)$$

Theorem 2.8 and Lemma 3.4 guarantee the existence of such a K_1.

Lemma 5.1. *Assume* (5)*; let γ be the circle of radius $2/\big(K_1L^{2p}\big)$ and of center* 0*, traveled once in the positive direction. Then, the spectrum of $\mathcal{D}$ does not intersect γ; the eigenprojection associated to the part of the spectrum contained inside γ is of rank 3.*

Proof. From Lemma 2.6 and Theorem 2.8, we know that the spectrum of $\mathcal{A}^\flat$ inside γ contains only the semisimple double eigenvalue 0 and the simple eigenvalue $\lambda^\flat$. We know also that $\mu_2^\flat$ is the closest element of the remainder of the spectrum of $\mathcal{A}^\flat$ to γ. Define

$$d(\zeta) = \min\big(|\zeta|, |\zeta - \lambda^\flat|, |\zeta - \mu_2^\flat|\big).$$

We define a twice punctured disk D by

$$D^\flat = \{\zeta \in \mathbb{C} : |\zeta| < \mu_2^\flat, \zeta \neq 0, \zeta \neq \lambda^\flat\}.$$

Then we have the following estimate on the resolvent $\mathcal{R}^0(\zeta) = (\zeta - \mathcal{A}^\flat)^{-1}$ of $\mathcal{A}^\flat$:

$$\forall \zeta \in D^\flat, \quad \big\|\mathcal{R}^0(\zeta)\big\|_{\mathcal{L}} \leq 1/d(\zeta).$$

It will be convenient to embed $\mathcal{D}$ into a holomorphic family of operators $\mathcal{D}(c)$ defined by

$$\mathcal{D}(c) = \mathcal{A}^\flat + c\mathcal{B} + c^2\mathcal{C},$$

and to do the same type of calculations as in Chapter II of [28]; however, the results quoted therein cannot be directly applied, since our eigenvalues are not uniformly isolated with respect to the remainder of the spectrum.

The Neumann perturbation series for $\mathcal{D}(c)$ is the expression

$$\mathcal{R}(\zeta, c) = \sum_{j=0}^{\infty}\big((\zeta - \mathcal{A}^\flat)^{-1}(c\mathcal{B} + c^2\mathcal{C})\big)^j(\zeta - \mathcal{A}^\flat)^{-1}; \tag{5.9}$$

this series converges provided that

$$|c| \le \min\bigl(d(\zeta)/\bigl(\|\mathcal{B}\|_{\mathcal{L}} + \|\mathcal{C}\|_{\mathcal{L}}\bigr), 1\bigr). \tag{5.10}$$

We infer from (5.8) that there exist $K_2 > 0$ and k_2 such that

$$\|\mathcal{B}\|_{\mathcal{L}} + \|\mathcal{C}\|_{\mathcal{L}} \le K_2 L^{k_2};$$

without loss of generality, we may assume that for all the $L's$ that we consider, $K_2 L^{k_2}$ is larger than 1; thus the function $\mathcal{R}$ is holomorphic with values in $\mathcal{L}$ on the set

$$\Delta = \bigl\{(\zeta, c) : \zeta \in D^\flat, L^{k_2} K_2 \,|c| < d(\zeta)\bigr\}, \tag{5.11}$$

and the following estimate holds over Δ:

$$\|\mathcal{R}(\zeta, c)\|_{\mathcal{L}} \le \frac{1}{d(\zeta) - |c| \, K_2 L^{k_2}}. \tag{5.12}$$

If γ is the circle defined in the statement of the lemma, any element of γ satisfies for L large enough $d(\zeta) > 1/(K_1 L^{2p})$. Therefore, the circle γ is included in the resolvent set of $\mathcal{D}(c)$ provided that

$$K_1 K_2 L^{2p+k_2} \,|c| < 1.$$

This will hold for $c = \kappa$ and L large enough, and the first assertion of the lemma is proved.

Since $\mathcal{R}$ is holomorphic in a neighborhood of $c = 0$, it admits a Taylor series of the form

$$\mathcal{R}(\zeta, c) = \sum_{j=0}^{\infty} \mathcal{R}^j(\zeta) c^j,$$

with

$$\mathcal{R}^j(\zeta) = \frac{1}{2\pi i} \int_{\gamma'} \mathcal{R}(\zeta, c) c^{-j-1} \, dc,$$

where γ' is a circle about 0 of radius strictly less than $d(\zeta)/\bigl(K_2 L^{k_2}\bigr)$. The first term of the expansion is equal to

$$\mathcal{R}^0(\zeta) = \bigl(\zeta - \mathcal{A}^\flat\bigr)^{-1}, \tag{5.13}$$

hence our notations are coherent. If we take γ' of radius $d(\zeta)/\bigl(2K_2 L^{k_2}\bigr)$, we have the estimates

$$\bigl\|\mathcal{R}^j(\zeta)\bigr\|_{\mathcal{L}} \le \frac{2}{d(\zeta)} \left(\frac{2K_2 L^{k_2}}{d(\zeta)}\right)^j, \tag{5.14}$$

and also, for all c such that $|c| < d(\zeta)/\bigl(2K_2 L^{k_2}\bigr)$, and for all $k \ge 0$,

$$\left\|\mathcal{R}(\zeta, c) - \sum_{j=0}^{k-1} \mathcal{R}^j(\zeta) c^j\right\|_{\mathcal{L}} \le \left[\frac{2K_2 L^{k_2} |c|}{d(\zeta)}\right]^k \frac{1}{d(\zeta) - 2K_2 L^{k_2} |c|}. \tag{5.15}$$

The projection on the total eigenspace of $\mathcal{D}(c)$ relative to the eigenvalues inside γ is given by

$$\mathcal{P}(c) = \frac{1}{2i\pi} \int_{\gamma} \mathcal{R}(\zeta, c) \, d\zeta.$$

We infer from (5.14) that the operator $\mathcal{P}^j$ defined by

$$\mathcal{P}^j = \frac{1}{2i\pi} \int_{\gamma} \mathcal{R}^j(\zeta) \, d\zeta$$

satisfies the estimate

$$\left\|\mathcal{P}^j\right\|_{\mathcal{L}} \le 4\left(4K_1K_2L^{2p+k_2}\right)^j;$$

for all c such that $2K_1K_2L^{k_2+2p}|c| < 1$ and for all $k \ge 0$, we have

$$\left\|\mathcal{P}(c) - \sum_{j=0}^{k-1}\mathcal{P}^j c^j\right\|_{\mathcal{L}} \le \frac{4\left(2K_1K_2L^{k_2+2p}|c|\right)^k}{1-2\sqrt{\kappa}K_1K_2L^{k_2+2p}}. \tag{5.16}$$

For $2K_1K_2L^{k_2+2p}|c| < 1$, $\mathcal{P}(c)$ is a holomorphic function of c. For $c = 0$, relation (5.13) implies that $\mathcal{P}_0$ is the projection on the space spanned by s, ir and r'; therefore, it is of dimension 3. A classical argument shows that a continuous family of projections which is of finite rank for some value of the parameter is of finite rank for all its values. This proves the second assertion of the lemma. □

Let us define now a basis of $\operatorname{Im}\mathcal{P}(c)$ which depends analytically on c in a neighborhood of 0; since we expect to find a nontrivial Jordan block, we will not look for a basis of eigenvectors of $\mathcal{D}(c)$.

Lemma 5.2. *Define vectors*

$$\begin{aligned} w(c) &= \mathcal{P}(c)\begin{pmatrix} s^\flat \\ 0 \end{pmatrix}, \quad t(c) = \mathcal{P}(c)\begin{pmatrix} -c\eta\sqrt{\varepsilon/\kappa} \\ r^\flat + c^2(\xi - r^\flat)/\kappa \end{pmatrix}, \\ v(c) &= \mathcal{P}(c)\begin{pmatrix} (r^\flat)' + c^2(\xi' - (r^\flat)')/\kappa \\ c\eta'\sqrt{\varepsilon/\kappa} \end{pmatrix}. \end{aligned} \tag{5.17}$$

There exist positive numbers $K_3 > 2K_1K_2$, $k_3 > k_2 + 2p$, K_4 and k_4 such that for $K_3L^{k_3}|c| < 1$ the vectors $w(c)$, $t(c)$ and $v(c)$ are holomorphic in c and form a basis of $\operatorname{Im}\mathcal{P}(c)$; in this basis, the restriction of the operator $\mathcal{D}(c)$ to $\operatorname{Im}\mathcal{P}(c)$ has matrix $M(c)$; $M(c)$ is analytic in c and for $K_3L^{k_3}|c| < 1$, it admits the expansion

$$M(c) = \sum_{j=0}^{\infty} M^j c^j,$$

with the estimate

$$\left\|M^j\right\|_{\mathcal{L}(\mathbb{C}^3)} \le K_4L^{k_4}\left(2K_3L^{k_3}\right)^j. \tag{5.18}$$

Similarly, w, t and v admit expansions

$$w(c) = \sum_{j=0}^{\infty} w^j c^j, \quad t(c) = \sum_{j=0}^{\infty} t^j c^j, \quad v(c) = \sum_{j=0}^{\infty} v^j c^j$$

with the estimates

$$\max\left(\left|w^j\right|, \left|t^j\right|, \left|v^j\right|\right) \le K_4L^{k_4}\left(2K_3L^{k_3}\right)^j.$$

Finally, when $c = \sqrt{\kappa}$, $M(c)$ has the form

$$M\left(\sqrt{\kappa}\right) = \begin{pmatrix} M_{11}(\sqrt{\kappa}) & 0 & 0 \\ M_{21}(\sqrt{\kappa}) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Proof. It is clear that w, t and v are holomorphic functions of c if $2K_1K_2L^{k_2+2p}|c| < 1$. By construction, they belong to $\operatorname{Im}\mathcal{P}(c)$. For $c = 0$, $w(0)$, $t(0)$ and $v(0)$ are orthogonal vectors spanning $\mathcal{P}(0)$; therefore, there is a neighborhood of $c = 0$ for which $w(c)$, $t(c)$ and $v(c)$ constitute a basis of $\operatorname{Im}\mathcal{P}(c)$. Let $\langle , \rangle$ denote the $\mathbb{C}$-bilinear product on $L^2(\mathbb{R};\mathbb{C})$ defined as follows: if

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \text{ and } Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$$

belong to $L^2(\mathbb{R};\mathbb{C})^2$, we let

$$\langle X, Y\rangle = \int_{\mathbb{R}} \big(X_1(x)Y_1(x) + X_2(x)Y_2(x)\big)\,dx.$$

In order to estimate the size of the neighborhood of 0 where $w(c)$, $t(c)$ and $v(c)$ constitute a basis of $\operatorname{Im}\mathcal{P}(c)$, we introduce a quasi Gram matrix $\Gamma(c)$ defined by

$$\Gamma(c) = \begin{pmatrix} \langle w(c), w(c)\rangle & \langle w(c), t(c)\rangle & 0 \\ \langle t(c), w(c)\rangle & \langle t(c), t(c)\rangle & 0 \\ 0 & 0 & \langle v(c), v(c)\rangle \end{pmatrix}.$$

The terms $\langle w(c), v(c)\rangle$ and $\langle t(c), v(c)\rangle$ vanish because w and t are even functions, while v is odd. We also remark that $\Gamma(c)$ is symmetric, not hermitian, and that

$$\Gamma(0) = \begin{pmatrix} 3/8 + O\big(\sqrt{\nu^\flat}\,\big) & 0 & 0 \\ 0 & 3(L+y)/2 + O(1) & 0 \\ 0 & 0 & 3/8 + O\big(\sqrt{\nu^\flat}\,\big) \end{pmatrix}.$$

We infer from the hypothesis $|c| \le 1$ and from (4.22) that there exist $K_3 > 2K_1K_2$ and $k_3 \ge 2k_2 + p$ such that

$$\big\|\Gamma(0)^{-1}\big(\Gamma(c) - \Gamma(0)\big\|_{\mathcal{L}(\mathbb{R}^3)} \le |c|\, K_3 L^{k_3}. \tag{5.19}$$

Therefore, if $|c|\, K_3 L^{k_3} < 1$, $\Gamma(c)$ is invertible, its inverse is analytical in c and it satisfies the estimate

$$\big\|\Gamma(c)^{-1}\big\|_{\mathcal{L}(\mathbb{C}^3)} \le \frac{\big\|\Gamma(0)^{-1}\big\|_{\mathcal{L}(\mathbb{C}^3)}}{1 - |c| K_3 L^{k_3}}.$$

Denote by

$$\begin{bmatrix} w(c) & t(c) & v(c) \end{bmatrix}$$

the function from $\mathbb{R}$ to 3×3 complex matrices whose columns vectors are respectively equal to $w(c)$, $t(c)$ and $v(c)$. We define three vectors $\hat{w}(c)$, $\hat{t}(c)$ and $\hat{v}(c)$ by

$$\begin{bmatrix} \hat{w}(c) & \hat{t}(c) & \hat{v}(c) \end{bmatrix} = \begin{bmatrix} \mathcal{P}(c)w(c) & \mathcal{P}(c)t(c) & \mathcal{P}(c)v(c) \end{bmatrix} \Gamma(c)^{-1}$$

Then, for $K_3L^{k_3}|c| < 1$, $\hat{w}(c)$, $\hat{t}(c)$ and $\hat{v}(c)$ are holomorphic in c and they constitute a basis of $\operatorname{Im}\mathcal{P}(c)^T$ which is dual to the basis $w(c)$, $t(c)$, $v(c)$ of $\operatorname{Im}\mathcal{P}(c)$, i.e.

$$\int_{\mathbb{R}} \begin{bmatrix} w(c) & t(c) & v(c) \end{bmatrix}^T \begin{bmatrix} \hat{w}(c) & \hat{t}(c) & \hat{v}(c) \end{bmatrix} dx = \mathbf{1}_{\mathbb{C}^3}.$$

Thus, $M(c)$ is given by

$$M(c) = \int_{\mathbb{R}} \begin{bmatrix} \hat{w}(c) & \hat{t}(c) & \hat{v}(c) \end{bmatrix}^T \begin{bmatrix} \mathcal{D}(c)w(c) & \mathcal{D}(c)t(c) & \mathcal{D}(c)v(c) \end{bmatrix} dx.$$

It is immediate that $\mathcal{D}(c)w(c)$, $\mathcal{D}(c)t(c)$ and $\mathcal{D}(c)v(c)$ are analytical in c with values in $L^2(\mathbb{R};\mathbb{C})^2$: in fact, they are polynomial in c. The study of section 4 and in particular estimates (4.22), completed by estimate (5.16) for $k = 0$ and estimate (5.19) show that there exist numbers K_4 and k_4 such that the following estimates hold:

$$\begin{gathered} \text{if } |c| = 1/\big(2K_3L^{k_3}\big), \text{ then} \\ \max\big(|w(c)|, |t(c)|, |v(c)|, |\hat{w}(c)|, \big|\hat{t}(c)\big|, |\hat{v}(c)|\big) \le K_4 L^{k_4} \\ \text{and } \max_{1\le i,j\le 3} |M_{jk}(c)| \le K_4 L^{k_4}. \end{gathered}$$

Standard methods of analytic functions give (5.18) and analogous estimates for y, t and v. Finally, differentiating with respect to β the relations

$$F\big(e^{i\beta}\big(\xi + i\sqrt{\varepsilon}\,\eta\big), \sqrt{\varepsilon}\,\tau, \sqrt{\varepsilon}, \nu\big) = 0 F\big(\big(\xi + i\sqrt{\varepsilon}\,\eta\big(\,\cdot + \beta\big), \sqrt{\varepsilon}\,\tau, \sqrt{\varepsilon}, \nu\big) = 0$$

we find that $t(\sqrt{\kappa})$ and $v(\sqrt{\kappa})$ are eigenvectors of $\mathcal{D}(\sqrt{\kappa})$ relatively to the eigenvalue 0. Thus w and t belong to $\operatorname{Im}\mathcal{P}$. Therefore the form of the matrix $M(\sqrt{\kappa})$ is clear, and this concludes the proof of the lemma. □

All our effort will be concentrated now on getting the first three terms of the expansion of $M_{11}(c)$ since we infer from (5.18) that

$$\left|M_{11}(\sqrt{\kappa}) - M_{11}^0 - \sqrt{\kappa}\, M_{11}^1 - \kappa M_{11}^2\right| \leq \kappa^{3/2}\mathcal{O}(1).$$

Lemma 5.3. *The first terms of the expansion of $\mathcal{P}$ have the form*

$$\mathcal{P}^1 = \begin{pmatrix} 0 & \mathcal{P}_{12}^1 \\ \mathcal{P}_{21}^1 & 0 \end{pmatrix}, \quad \mathcal{P}^2 = \begin{pmatrix} \mathcal{P}_{11}^2 & 0 \\ 0 & \mathcal{P}_{22}^2 \end{pmatrix}, \tag{5.20}$$

and the first terms of the expansion of w and t are given by

$$w^1 = \begin{pmatrix} 0 \\ s_1 \end{pmatrix}, \quad w^2 = \begin{pmatrix} s_2 \\ 0 \end{pmatrix}, \quad t^1 = \begin{pmatrix} r_1 \\ 0 \end{pmatrix}, \quad t^2 = \begin{pmatrix} 0 \\ r_2 \end{pmatrix}, \tag{5.21}$$

with

$$s_1 = (\lambda^\flat - B^\flat)^{-1} Q_\perp^\flat \mathcal{B}_{21} s^\flat \text{ and } r_1 = -(A^\flat)^{-1} P_\perp^\flat \mathcal{B}_{12} r^\flat - \sqrt{\frac{\varepsilon}{\kappa}} P^\flat \eta. \tag{5.22}$$

Proof. The first statement of the lemma is an immediate consequence of the definition of the $\mathcal{P}^j$'s and of the special form of $\mathcal{A}^\flat$, $\mathcal{B}$ and $\mathcal{C}$. Let us calculate $\mathcal{P}_{21}^1 s^\flat$: $\mathcal{R}^1(\zeta)$ is equal to $\mathcal{R}^0(\zeta)\mathcal{B}\mathcal{R}^0(\zeta)$ and thus

$$\mathcal{R}^1(\zeta)\begin{pmatrix} s^\flat \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ (\lambda^\flat - \zeta)^{-1}(\zeta - B^\flat)^{-1}\mathcal{B}_{21}s^\flat \end{pmatrix}.$$

Hence, thanks to the theorem of residues,

$$s_1 = \mathcal{P}_{21}^1 s^\flat = \frac{1}{2i\pi}\int_\gamma (\zeta - B^\flat)^{-1}\mathcal{B}_{21}s^\flat \frac{d\zeta}{\lambda^\flat - \zeta} = (\lambda^\flat - B^\flat)^{-1} Q_\perp^\flat \mathcal{B}_{21} s^\flat.$$

Similarly, according to the definition of $t(c)$,

$$r_1 = \mathcal{P}_{12} r^\flat - \sqrt{\frac{\varepsilon}{\kappa}} P^\flat \eta,$$

and

$$\mathcal{P}_{12}^1 r^\flat = \frac{1}{i\pi}\int_\gamma (\zeta - A^\flat)^{-1}\mathcal{B}_{12} r^\flat \frac{d\zeta}{\zeta} = (-A^\flat)^{-1} P_\perp^\flat \mathcal{B}_{12} r^\flat.$$

□

Now we are able to give expressions for the first coefficients of the power expansion of $M_{11}(c)$:

Lemma 5.4. *The coefficients M_{11}^j, for $0 \leq j \leq 2$ and M_{21}^j for $j = 0, 1$ are given by*

$$M_{11}^0 = \lambda^\flat, \quad M_{21}^0 = 0, \tag{5.23}$$

$$M_{11}^1 = 0, \quad M_{21}^1 = \frac{(\mathcal{B}_{21}s^\flat - \lambda^\flat s_1, r^\flat)}{|r^\flat|^2}, \tag{5.24}$$

$$M_{11}^2 = \frac{(\mathcal{B}_{12}s_1 + \mathcal{C}_{11}s^\flat - M_{21}^1 r_1, s^\flat)}{|s^\flat|^2}. \tag{5.25}$$

Proof. From the analyticity properties we infer that

$$\mathcal{A}^\flat y^0 = M_{11}^0 y^0 + M_{21}^0 t^0, \tag{5.26}$$

$$\mathcal{A}^\flat w^1 + \mathcal{B} y^0 = \sum_{j=0}^{1} \big(M_{11}^j y^{1-j} + M_{21}^j t^{1-j} \big) \tag{5.27}$$

$$\mathcal{A}^\flat w^2 + \mathcal{B} w^1 + \mathcal{C} y^0 = \sum_{j=0}^{2} \big(M_{11}^j y^{2-j} + M_{21}^j t^{2-j} \big) \tag{5.28}$$

From (5.26) and the value of $t(0)$ we infer immediately (5.23). Relation (5.24) can be rewritten with the help of (5.21) as

$$\begin{pmatrix} 0 \\ B^\flat r_1 \end{pmatrix} + \begin{pmatrix} 0 \\ \mathcal{B}_{21} s^\flat \end{pmatrix} = \lambda^\flat \begin{pmatrix} 0 \\ s_1 \end{pmatrix} + M_{11}^1 \begin{pmatrix} s^\flat \\ 0 \end{pmatrix} + M_{21}^1 \begin{pmatrix} 0 \\ r^\flat \end{pmatrix}.$$

It is immediate that M_{11}^1 vanishes. Moreover, we perform the scalar product of the second component of the above identity with $r^\flat$; since $(B^\flat s_1, r^\flat) = (s_1, B^\flat r^\flat) = 0$, we obtain the second relation of (5.24).

Finally, (5.28) implies that

$$A^\flat s_2 + \mathcal{B}_{12} s_1 + \mathcal{C}_{11} s^\flat = \lambda^\flat s_2 + M_{11}^2 s^\flat + M_{21}^1 r_1.$$

If we perform the scalar product of this relation with $s^\flat$ and if we observe that $(A^\flat s_2, s^\flat) = (s_2, \lambda^\flat s^\flat)$, we obtain (5.25). □

We give now an asymptotic of M_{11}^2 in terms of $r^\flat$, $q^\flat$ and $\sigma^\flat$:

Theorem 5.5. *The coefficient M_{11}^2 has the following asymptotics*

$$M_{11}^2 = \frac{\varepsilon}{\kappa} |s^\flat|^{-2} \left[\frac{\partial a}{\partial L} \right]^\flat + \big(\sqrt{\nu^\flat} + \sqrt{\kappa} \big) \mathcal{O}(1), \tag{5.29}$$

where a has been defined at (4.10).

Proof. According to (2.35) and Lemma 4.4, we have

$$\mathcal{B}_{21} s^\flat = \sqrt{\frac{\varepsilon}{\kappa}} \big[(\theta - 3r^2 - 2rq + 4r^3 q) \sigma \big]^\flat + (\kappa + \nu^\flat) \mathcal{O}(1),$$

and in virtue of (3.25) and (3.26), we can see that

$$\mathcal{B}_{21} s^\flat = -\sqrt{\frac{\varepsilon}{\kappa}} (B q_1 + \theta_1 r)^\flat + (\kappa + \nu^\flat) \mathcal{O}(1). \tag{5.30}$$

Therefore

$$s_1 = \sqrt{\frac{\varepsilon}{\kappa}} Q_\perp^\flat q_1^\flat + (\kappa + \nu^\flat) \mathcal{O}(1) \tag{5.31}$$

and

$$M_{21}^1 = -\sqrt{\frac{\varepsilon}{\kappa}} \theta_1^\flat + (\kappa + \nu^\flat) \mathcal{O}(1).$$

Similarly, according to Lemma 4.4,

$$\mathcal{B}_{12} r^\flat = \sqrt{\frac{\varepsilon}{\kappa}} \big[(-\theta + r^2 - 2rq + 4r^3 q) r \big]^\flat + \kappa \mathcal{O}(1),$$

and thanks to (3.5) we have

$$\mathcal{B}_{12} r^\flat = \sqrt{\frac{\varepsilon}{\kappa}} (A q)^\flat + (\kappa + \nu^\flat) \mathcal{O}(1).$$

Therefore

$$r_1 = -\sqrt{\frac{\varepsilon}{\kappa}}q^\flat + (\kappa + \nu^\flat)\mathcal{O}(1),$$

and in particular,

$$M_{21}^1(r_1, s^\flat) = \frac{\varepsilon}{\kappa}\theta_1^\flat(q, \sigma)^\flat + (\kappa + \nu^\flat)\mathcal{O}(1). \tag{5.32}$$

There remains to calculate

$$(\mathcal{B}_{12}s_1 + \mathcal{C}_{11}s^\flat, s^\flat).$$

We infer from (5.31) that

$$(\mathcal{B}_{12}s_1, s^\flat) = \left(\mathcal{B}_{12}\sqrt{\frac{\varepsilon}{\kappa}}q_1^\flat, s^\flat\right) - \left(\mathcal{B}_{12}\sqrt{\frac{\varepsilon}{\kappa}}Q^\flat q_1^\flat, s^\flat\right) + (\kappa + \nu^\flat)\mathcal{O}(1)$$

and we have the following relation

$$(\mathcal{B}_{12}r^\flat, s^\flat) = \sqrt{\frac{\varepsilon}{\kappa}}\big((Aq)^\flat + \kappa\mathcal{O}(1), s^\flat\big) = (\kappa + \nu^\flat)\mathcal{O}(1).$$

Therefore

$$\begin{aligned}(\mathcal{B}_{12}s_1, s^\flat) &= \sqrt{\frac{\varepsilon}{\kappa}}(\mathcal{B}_{12}q_1^\flat, s^\flat) + (\kappa + \nu^\flat)\mathcal{O}(1)\\ &= \frac{\varepsilon}{\kappa}\big((-\theta + r^2 - 2rq + 4r^3q)q_1, \sigma\big)^\flat + (\kappa + \nu^\flat)\mathcal{O}(1).\end{aligned}$$

On the other hand,

$$\begin{aligned}(\mathcal{C}_{11}s^\flat, s^\flat) &= \big((5r^4 - 3r^2)\sigma, \sigma\big)^\flat + \frac{1}{\kappa}\big([5\xi^4 - 3\xi^2 - (5r^4 - 3r^2)^\flat]\sigma^\flat, \sigma^\flat\big)\\ &\quad + \frac{\varepsilon}{\kappa}\big((2rq - q^2 + 6r^2q^2)\sigma, \sigma\big)^\flat + \kappa\mathcal{O}(1).\end{aligned}$$

But according to (4.22), a Taylor expansion gives

$$5\xi^4 - 3\xi^2 - (5r^4 - 3r^2)^\flat = (20r^3 - 6r)^\flat(\xi - r^\flat) + \kappa^2\mathcal{O}(1)$$

and thanks to Lemma 4.4, this expression is equal to $-(20r^3 - 6r)\xi_1 + \kappa^2\mathcal{O}(1)$. Hence, from (2.42)

(5.33)

$$\begin{aligned}\big([5\xi^4 - 3\xi^2 - (5r^4 - 3r^2)^\flat]\sigma^\flat, \sigma^\flat\big) &= (A^\flat\sigma_2^\flat, \xi_1) + \kappa\nu^\flat\mathcal{O}(1)\\ &= \big(\sigma_2^\flat, G_1(U^\flat, \nu)\big) + \kappa\nu^\flat\mathcal{O}(1)\\ &= \big(\sigma_2^\flat, \kappa(r^5 - r^3)^\flat + \varepsilon(r^2q + 2r^3q - rq^2 - \theta q)^\flat\big)\\ &\qquad + \kappa(\nu^\flat + \kappa)\mathcal{O}(1).\end{aligned}$$

Now we obtain the expansion

$$\begin{aligned}&(\mathcal{C}_{11}s^\flat, s^\flat) + (\mathcal{B}_{12}s_1, s^\flat)\\ &= \big((5r^4 - 3r^2)\sigma, \sigma\big)^\flat + \frac{\varepsilon}{\kappa}\big((-\theta + r^2 - 2rq + 4r^3q)q_1, \sigma\big)^\flat + (r^5 - r^3, \sigma_2)^\flat\\ &+ \frac{\varepsilon}{\kappa}(r^2q + 2r^3q^2 - rq^2 - \theta q, \sigma_2)^\flat + \frac{\varepsilon}{\kappa}\big((2rq - q^2 + 6r^2q^2)\sigma, \sigma\big)^\flat + (\kappa + \nu^\flat)\mathcal{O}(1).\end{aligned}$$

We observe that

$$\begin{aligned}&\big((5r^4-3r^2)\sigma,\sigma\big)+(\sigma_2,r^5-r^3)\\&\quad=2\int_0^\infty\big(\sigma_2(r^5-r^3)+\sigma^2(5r^4-3r^2)\big)\,dx\\&=-2\int_0^\infty(r^5-r^3)(r_{xx}+2\sigma_x)\,dx+2\int_0^\infty r_x^2(5r^4-3r^2)\,dx+\sqrt{\nu}\mathcal{O}(1)\\&=\sqrt{\nu}\,\mathcal{O}(1);\end{aligned}$$

therefore, with the help of (5.32), we are left with

$$\left|s^\flat\right|^2M_{11}^2=\frac{\varepsilon}{\kappa}\left[\frac{\partial}{\partial L}(r^2q+2r^3q^2-rq^2-\theta q,\sigma)\right]^\flat+(\sqrt{\kappa}+\sqrt{\nu^\flat})\mathcal{O}(1).$$

This concludes the proof of the theorem. □

Therefore, the only remaining question is to obtain an asymptotic for $\partial a_1/\partial L$.

Theorem 5.6. *The following asymptotics holds:*

$$\frac{\partial a}{\partial L}=\frac{3\pi^2}{32L^2}+O\big(L^{-5/2}\big).$$

Proof. We recall (4.12); we expect that the dominant term in a will be

$$b=m\frac{\partial}{\partial L}(r\sigma\phi',\phi'),$$

the remaining terms being small with respect to b, as we will check later. The number b is also equal to

$$b=m\int\left(\sigma^2+r\sigma_2\right)|\phi'|^2\,dx+2m\int r\sigma\phi'\phi_1'\,dx.$$

Thanks to estimate (2.23),

$$\int\left(\sigma^2+r\sigma_2\right)|\phi'|^2\,dx=2\int_0^\infty\left[\sigma^2+r\big(-2\sigma'-r''\big)\right]|\phi'|^2\,dx+O\big(\sqrt{\nu}\,\big),$$

and by integration by parts:

$$\int\left(\sigma^2+r\sigma_2\right)|\phi'|^2\,dx=2\int_0^\infty\left[\big(\sigma+r'\big)^2|\phi'|^2+2r(2\sigma+r')\phi'\phi_b''igr\right]dx+O\big(\sqrt{\nu}\big).$$

Now, thanks to (2.22), we obtain

$$b=4m\int_0^\infty r\sigma\phi'\left(\phi''+\phi_1'\right)dx+\sqrt{\nu}\mathcal{O}(1).$$

We differentiate (3.13) with respect to L, and we recall the definition (2.20) of S; then

$$\begin{aligned}m\big(\phi''+\phi_1'\big)&=-\frac{\theta_1}{R}\int_x^\infty R\,dy\\&-\phi'\frac{\partial m}{\partial L}+\frac{1}{R^2}(S+R')\int_x^\infty(\theta-R)R\,dy-\frac{1}{R(x)}\int_x^\infty(\theta-2R)(S+R')\,dx\end{aligned}$$

In the above expression, the principal term is the first term on the right hand side; the other terms are estimated as follows: it is immediate that

$$\phi'\partial m/\partial L=\nu O(1);$$

thanks to (2.21) and (3.20),

$$\frac{S+R'}{R^2}\int_x^\infty(\theta-R)R\,dx=O\big(\sqrt{\nu}\,\big);$$

thanks to (2.21), (3.19) and (3.21), we also have

$$\left|\frac{1}{R(x)}\int_x^\infty (\theta-2R)(S+R')\,dx\right| \le \frac{O(1)\nu}{\tilde R(x-L)}\int_{x-L}^\infty \frac{\tilde R(y)}{1-4\tilde R(y)/3}\,dy = O(\nu);$$

therefore,

$$m\big(\phi''+\phi_1'\big) = -\frac{\theta_1}{R}\int_x^\infty R\,dy + O\big(\sqrt{\nu}\,\big).$$

Thanks to (2.11) and (2.22),

$$\sigma r = -\frac{1}{2}\tilde R'(\cdot-L) + O(1)\nu^{3/4}e^{-2(x-L)};$$

it is straightforward that

$$\int_0^\infty \nu^{3/4}e^{-(2x-2L)}\frac{\ln(1-4\tilde R(x-L)/3)}{\tilde R(x-L)}\,dx = O\big(\nu^{1/4}L\big)$$

and

$$\int_0^\infty O\big(\sqrt{\nu}\,\big)\phi' r\sigma\,dx = O\big(\sqrt{\nu}\,\big).$$

Thus, we have proved that

$$b = -\frac{3\theta_1}{4}\int_0^\infty \tilde R'(\cdot-L)\frac{\ln(1-4\tilde R(\cdot-L)/3)}{\tilde R(\cdot-L)}\,\phi'\,dx + O\big(L\nu^{1/4}\big).$$

In order to give an asymptotic of the integral in the above expression, we cut it into three pieces: one piece from 0 to $L-\sqrt{L}$, which is

$$\int_0^{L-\sqrt{L}} \frac{\ln(1-4\tilde R(\cdot-L)/3)}{\tilde R(\cdot-L)}\tilde R'(\cdot-L)O(1)\,dx = L^2O(1)e^{-2\sqrt{L}};$$

the second piece is

$$2\int_{L-\sqrt{L}}^\infty \frac{\ln(1-4\tilde R(\cdot-L)/3)}{\tilde R(\cdot-L)}\tilde R'(\cdot-L)\,dx$$

which we integrate thanks to the change of variable $y = 4\tilde R(x-L)/3$: it is thus equal to

$$2\int_0^{\tilde R(-\sqrt{L})}\frac{\ln(1-y)}{y}\,dy = -\frac{\pi^2}{3} + \sqrt{L}O\big(e^{-\sqrt{L}}\big).$$

The last piece is

$$\int_{L-\sqrt{L}} (\phi'(x)+2)\frac{\ln(1-4\tilde R(\cdot-L)/3)}{\tilde R(\cdot-L)}\tilde R'(\cdot-L)\,dx;$$

since for $x \ge L-\sqrt{L}$, $\phi'+2 = O\big(1/\sqrt{L}\,\big)$, this last piece is an $O(1/\sqrt{L})$. Finally, we have obtained

$$b = \frac{3\pi^2}{32L^2} + O\big(L^{-5/2}\big).$$

There remains to estimate the other terms in $\partial a/\partial L$. It is easy to see that all of them are of order $\nu\mathcal{O}(1)$, and therefore negligible before the error given in the above formula. This completes the proof of the Theorem. □

We are now in grade to state the stabilization property:

Theorem 5.7. *For $0 \leq y \leq L^p$, the following asymptotic holds:*

$$M_{11}\big(\sqrt{\kappa}\big) = -\frac{3\nu^\flat}{2} + \frac{\varepsilon\pi^2}{4(L+y)^2} + \kappa O(L^{-5/2}) + (\nu^\flat)^{3/2}\mathcal{O}(1). \tag{5.34}$$

There is a critical y_c such that $a(y_c, \nu)$ vanishes; an equivalent for y_c is given by

$$y_c \sim \frac{1}{2}\ln L.$$

The corresponding critical parameter is

$$\alpha_c(\nu) = \frac{1}{2}\sqrt{\nu}\left(1 - \frac{\pi^2}{48L^2} + O\big(L^{-5/2}\big)\right). \tag{5.35}$$

Proof. The asymptotic (5.34) is a consequence of all the previous results. In order to find the critical value of y, we have to solve the equation

$$-\frac{3}{2}\nu^\flat + \frac{\varepsilon\pi^2}{4(L+y)^2} + \big(\nu^\flat\big)^{3/2}\mathcal{O}(1) + \kappa O\big(L^{-5/2}\big) = 0.$$

After replacing κ and $\nu^\flat$ by their respective values, and defining the new unknown

$$Y = e^{-4Y},$$

we find the following equation in Y:

$$-Y + \frac{\pi^2(1-Y)}{24(L - \ln Y/4)^2} + O\big(L^{-5/2}\big) = 0$$

It is clear that there is a solution Y_c of this equation which satisfies

$$Y_c = \frac{\pi^2}{24L^2} + O\big(L^{-5/2}\big).$$

Using relations (4.4), (4.15), (4.22) and (5.34), we can see now that at the critical value of y,

$$\varepsilon = \frac{\nu}{4}\left(1 - \frac{\pi^2}{48L^2} + O\big(L^{-5/2}\big)\right).$$

Relation (5.35) is an immediate consequence of the above, together with $\alpha = \sqrt{\varepsilon}$. □

The final result of this section is

Proposition 5.8. *Let u be the solution defined at theorem 4.5. Let $|\alpha| \leq \alpha_m$ be the number defined at (4.28) and α_c be the number defined at (5.35). Then, the solution u is stable iff $\alpha_c < |\alpha| < \alpha_m$, and unstable if $|\alpha| \leq \alpha_c$. More precisely, in the first case, the spectrum of the linearized operator at u contains exactly one negative eigenvalue, if $|\alpha| < \alpha_c$; when $_alpha = \alpha_c$, the eigenvalue 0 is of algebraic multiplicity 3 and geometric multiplicity 2, with a non trivial Jordan block of dimension 2; in the second case, the linearized operator at u has the semisimple double eigenvalue 0, and the remainder of the spectrum is included in $\Re z \geq \mu > 0$.*

Proof. This Proposition summarizes our previous analysis: the part of the spectrum of $\mathcal{D}$ which is outside of the disk of radius L^{-2p-1} around 0 is contained in the half plane

$$\Re\zeta \geq CL^{-2p},$$

for ν small enough, according to Theorem 2.8, Lemma 3.4 and the definition (5.2) of $\mathcal{D}$. Therefore, the change of stability is equivalent to the change of sign of $M_{11}(\sqrt{\kappa})$ given by (5.34). Theorem 5.7 implies that M_{11} vanishes for some value α_c of $|\alpha|$, whose asymptotic is given by (5.35). The results on the multiplicity of 0 will be clear provided that we show that M_{21} does not vanish; but formula (5.30)

together with (3.8) proves that this is the case. This concludes the proof of our proposition. ☐

The last step is to understand the evolution of small perturbations of the pulse; the linearized evolution is easy to understand: if $\delta_i(t)$ are the coordinates in the basis $\{z, iu, u'\}$ of the projection of the linearized perturbation on the space $\operatorname{Im} Q$, then

$$\begin{pmatrix}\delta_1(t)\\ \delta_2(t)\\ \delta_3(t)\end{pmatrix} = e^{i\omega t}\begin{pmatrix} e^{-at} & 0 & 0\\ b(e^{-at}-1)/a & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\delta_1(0)\\ \delta_2(0)\\ \delta_3(0)\end{pmatrix}..$$

For $a > 0$, we can see that the asymptotic behavior of the perturbation is simply described by a change of phase. The smaller a, the larger this change of phase, and the smaller the convergence to the phase shift.

This linear theory can be justified, using [18], Chapter 5, Exercise 6. Let us denote by $u(x,t)$ the stable pulse that we have found, and by $\tilde{u}(x,t)$ the perturbed solution of (1.7). It is clear that $\mathcal{B}$ is a sectorial operator, because it is a bounded perturbation of a positive self-adjoint operator. Then there exists in the stable case a $\delta > 0$ such that if $|\tilde{u}_0 - u(\cdot,0)|_{H^2(\mathbb{R})} \le \delta$, then, there exist $X(\tilde{u}_0)$ and $\Psi(\tilde{u}_0)$ such that

$$\lim_{t\to\infty} |\tilde{u}(x,t) - u(x - X(\tilde{u}_0), t)e^{i\Psi(\tilde{u}_0)}|_{H^2(\mathbb{R})} = 0.$$

Moreover, a straightforward calculation shows that (5) gives the correct behavior for small δ.

6. The general situation

It is indeed possible to perform the same kind of computations for the general case, i.e. equation (1.8), under the assumption

$$\Im m_j = \sqrt{\varepsilon}\,\mu_j, \quad j = 1, 2, 3. \tag{6.1}$$

let us give the main steps of the calculation; the functional validation is entirely identical, and we will shorten the exposition by giving only the relevant part of the expansion, and without bothering to give estimates on the remainders.

Under assumption (6.1), the functional F becomes

$$F(u) = i\omega u - mu'' + \big(m + i\sqrt{\varepsilon}\,\mu_1\big)u - \big(1 + i\sqrt{\varepsilon}\,\mu_2\big)|u|^2 u + \big(1 + i\sqrt{\varepsilon}\,\mu_3\big)\,|u|^4\,u.$$

The functions G_1 and G_2, which are analogous to the ones defined at the beginning of section 3 are given by

$$\begin{aligned} G_1(\xi,\eta,\tau,\varepsilon,\nu) &= -\varepsilon\tau\eta - m\xi'' + m\xi - \mu_1\varepsilon\eta - \big(\xi^2 + \varepsilon\eta^2\big)\big(\xi - \mu_2\varepsilon\eta\big)\\ &\quad + \big(\xi^2 + \varepsilon\eta^2\big)^2\big(\xi - \mu_3\varepsilon\eta\big),\\ G_2(\xi,\eta,\tau,\varepsilon,\nu) &= \tau\xi - m\eta'' + m\eta + \mu_1\xi - \big(\mu_2\xi + \eta\big)\big(\xi^2 + \varepsilon\eta^2\big)\\ &\quad + \big(\mu_3\xi + \eta\big)\big(\xi^2 + \varepsilon\eta^2\big)^2. \end{aligned}$$

The second equation (3.5) is changed into

$$Bq = -\theta r - \mu_1 r + \mu_2 r^3 - \mu_3 r^5, \quad (q, r) = 0,$$

and θ is determined by the orthogonality condition

$$\int \big((\theta + \mu_1)R - \mu_2 R^2 + \mu_3 R^3\big).$$

For ϕ given by (3.11), we have the relations

$$-m\big(r^2\phi'\big)' = -\theta r^2 - \mu_1 r^2 + \mu_2 r^4 - \mu_3 r^6$$

and

$$m\phi'(x) = \frac{1}{R(x)} \int_x^\infty \left[-\theta R - \mu_1 R + \mu_2 R^2 - \mu_3 R^3\right] dy.$$

The derivative q_1 of q with respect to L satisfies the relation

$$Bq_1 = -\theta_1 r - \theta\sigma - \mu_1\sigma + 3\mu_2 r^2\sigma - 5\mu_3 r^4\sigma + 2rq\sigma - 4r^3 q\sigma + \nu\mathcal{O}(1).$$

The asymptotics for θ, θ_1 and ϕ' are given by

$$\theta = -\mu_1 + \frac{3}{4}\mu_2 - \frac{9}{16}\mu_3 - \frac{3\mu_2}{8L} + \frac{27\mu_3}{64L} + \sqrt{\nu}\,O(1),$$
$$\theta_1 = \frac{3\mu_2}{8L^2} - \frac{27\mu_3}{64L^2} + \sqrt{\nu}\,O(1)$$

and

$$\phi'(x) \sim \frac{c_1}{L}\left(1 + e^{2(x-L)}\right)\ln\left(1 + e^{2(L-x)}\right) + c_2 + \frac{3\mu_3}{4}\left(1 + e^{2(x-L)}\right)^{-1} \tag{6.2}$$

where the numbers c_1, c_2 and c_3 are given by:

$$c_1 = \mu_2 - \frac{9\mu_3}{8}, \quad c_2 = -2\mu_2 + \frac{3\mu_3}{2}, \quad c_3 = \frac{3\mu_3}{4}. \tag{6.3}$$

We obtain these asymptotics with the help of the following calculations:

$$\int_x^\infty \tilde{R}(y)\,dy = \frac{3}{8}\ln\left(1 + e^{-2x}\right),$$
$$\int_x^\infty \tilde{R}(y)^2\,dy = \frac{9}{32}\left[\ln\left(1 + e^{-2x}\right) - \left(1 + e^{2x}\right)^{-1}\right],$$
$$\int_x^\infty \tilde{R}(y)^3\,dy = \frac{27}{128}\left[\ln\left(1 + e^{-2x}\right) - \left(1 + e^{2x}\right)^{-1} - \left(1 + e^{2x}\right)^{-2}/2\right].$$

The existence ansatz is completely analogous to the one defined at the beginning of section 4 by (4.1), (4.2) and (4.3); the definition of a_0 is as in (4.7), and we let

$$a_1 = \left((-\theta - \mu_1 + \mu_2 r^2 - \mu_3 r^4)q - q^2 r + 2r^3 q^2, s\right)$$
$$a_2 = \left(\mu_2 q^3 + rq^4 - 2\mu_3 r^2 q^3, s\right),$$
$$a_3 = -\left(\mu_3 q^5, s\right).$$

It is straightforward to prove that the new a_1 is still equivalent to $m\left(r\sigma\phi', \phi'\right)$ as in (4.12), and we just have to calculate $\phi'(L)$ in order to apply formula (4.13); we infer from estimate (6.2) that

$$a_1 \sim \frac{9}{64}\left(-2\mu_2 + 15\mu_3/8\right)^2 + O(1/L).$$

Therefore,

$$\varepsilon^\flat \sim \kappa\left(-2\mu_2 + 15\mu_3/8\right)^{-2}. \tag{6.4}$$

The remainder of the existence proof is identical to the proof given in section 4, its details are left to the reader.

Next step is to calculate $D_u F$ at the solution obtained by this existence proof; we will have (5.2), with $\mathcal{B}$ and $\mathcal{C}$ of the same form as in (5.3), and the new values

for the coefficients of ${}_calb$ and $\mathcal{C}$ are given by

$$\mathcal{B}_{12} = \frac{\sqrt{\varepsilon}\big[-\tau - \mu_1 - 2\xi\eta + |u|^2\mu_2 + \eta^2\varepsilon + 4|u|^2\xi\eta - |u|^4\mu_3 - 4|u|^2\eta^2\varepsilon\mu_3\big]}{\sqrt{\kappa}(1-\kappa)}$$

$$\mathcal{B}_{21} = \frac{\sqrt{\varepsilon}\big[\tau + \mu_1 - |u|^2 - 2\xi^2\mu_2 - 2\xi\eta + 4\xi\eta|u|^2 + \mu_3|u|^4 + 4\mu_3|u|^2\xi^2\big]}{\sqrt{\kappa}(1-\kappa)},$$

$$\mathcal{C}_{11} = \frac{1}{\kappa}\Big[\frac{-|u|^2 - 2\xi^2 + 2\varepsilon\mu_2\xi\eta + |u|^4 + 4|u|^2\xi^2 - 4\mu_3\xi\eta|u|^2\varepsilon}{1-\kappa} + \big(3r^2 - 5r^4\big)^\flat\Big],$$

$$\mathcal{C}_{22} = \frac{1}{\kappa}\Big[\frac{-|u|^2 - 2\eta^2\varepsilon - 2\eta\xi\mu_2\varepsilon + |u|^4 + 4|u|^2\eta^2\varepsilon + 4\mu_3|u|^2\eta\xi}{1-\kappa} + \big(r^2 - r^4\big)^\flat\Big].$$

The principal part of $\mathcal{B}_{12}$ is

$$\sqrt{\frac{\varepsilon}{\kappa}}\big(-\theta - \mu_1 - 2rq + \mu_2 r^2 + 4r^3 q - \mu_3 r^4\big)^\flat,$$

and the principal part of $\mathcal{B}_{21}$ is

$$\sqrt{\frac{\varepsilon}{\kappa}}\big(\theta + \mu_1 - 3\mu_2 r^2 - 2rq + 4r^3 q + 5\mu_3 r^4\big)^\flat.$$

The principal part of $\mathcal{C}_{11}$ is

$$\big(5r^4 - 3r^2\big)^\flat + \frac{\varepsilon}{\kappa}\big(-r^2 + 2\mu_2 rq + 6r^2q^2 - 4r^3q\mu_3\big)^\flat + \frac{5\xi^4 - 3\xi^2 - \big(5r^4 - 3r^2\big)^\flat}{\kappa}.$$

Lemmas 5.1, 5.2 and 5.3 have an identical statement and an identical proof. The calculations of Lemma 5.4 are modified only to get M^2_{11}: we have first

$$\begin{aligned}&\big(\mathcal{C}_{11}s^\flat, s^\flat\big) \sim \big(\mathcal{C}_{11}\sigma^\flat, \sigma^\flat\big)\\ &\sim \big((5r^4 - 3r^2)\sigma, \sigma\big)^\flat + \frac{\varepsilon}{\kappa}\big((-r^2 + 2\mu_2 rq + 6r^2q^2 - 4r^3q\mu_3)\sigma, \sigma\big)^\flat\\ &\quad + \frac{\big(5\xi^4 - 3\xi^2 - (5r^4 - 3r^2)^\flat\sigma^\flat, \sigma^\flat\big)}{\kappa}.\end{aligned}$$

With the same computation as in (5.33), the last term of the principal part is $\big(\sigma_2^\flat, G_1(U^\flat, \nu)\big)/\kappa$ which is equal to

$$\big(\sigma_2, r^5 - r^3\big)^\flat + \frac{\varepsilon}{\kappa}\big(\sigma_2, -\theta q - \mu_1 q - rq^2 + \mu_2 r^3 q + 2r^3q^2 - \mu_3 r^4 q\big)^\flat,$$

up to higher order terms. On the other hand

$$\big(\mathcal{B}_{12}s_1, s^\flat\big) \sim \sqrt{\frac{\varepsilon}{\kappa}}\big(\mathcal{B}_{12}q_1^\flat, s^\flat\big) \sim \frac{\varepsilon}{\kappa}\big((-\theta - \mu_1 - 2rq + \mu_2 r^2 + 4r^3q - \mu_3 r^4)q_1, \sigma\big)^\flat.$$

We find finally the same result as in (5.29).

The computation of $\partial a/\partial L$ proceeds along the lines of Theorem 5.6. The principal part of $\partial a/\partial L$ is

$$4m\int_0^\infty r\sigma\phi'\big(\phi'' + \phi_1'\big)\,dx$$

and

$$m\big(\phi'' + \phi_1'\big) \sim -\frac{\theta_1}{R(x)}\int_x^\infty R\,dy.$$

Therefore, it suffices to find an equivalent of

$$2\theta_1\int_0^\infty \frac{\tilde{R}'(x-L)\phi'(x)}{\tilde{R}(x-L)}\int_x^\infty \tilde{R}(y-L)\,dy\,dx.$$

We replace ϕ' by its expansion (6.2) which implies that the expression that we wish to estimate is

$$-\frac{3\theta_1}{4}\left[\frac{c_1}{L}\int_{-L}^{\infty} 2e^{2x}\left[\ln\left(1+e^{-2x}\right)\right]^2 dx + c_2\int_{-L}^{\infty}\frac{2e^{2x}\ln\left(1+e^{-2x}\right)}{1+e^{2x}}\,dx\right.$$
$$\left.+c_3\int_{-L}^{\infty}\frac{2e^{2x}\ln\left(1+e^{-2x}\right)}{\left(1+e^{2x}\right)^2}\,dx\right].$$

Each of the three integrals is equal, up to terms of order $O\left(\sqrt{\nu}\right)$ to the following values:

$$\pi^2/3, \quad \pi^2/6, \quad 1;$$

the computation is straightforward, and uses the change of variable

$$1+e^{2x}=\frac{1}{1-y}.$$

All these calculations lead to the following principal part:

$$\frac{\partial a}{\partial L}=\frac{1}{L^2}\left[\frac{3\mu_2}{8}-\frac{27\mu_3}{64}\right]\left[\frac{\pi^2\mu_2}{4}-\frac{3\pi^2\mu_3}{16}+\frac{9\mu_3}{16}\right]+O\left(L^{-5/2}\right).$$

Finally, we find that

$$M_{11}\left(\sqrt{\kappa}\right)=-\lambda^\flat+\frac{\kappa\chi(\mu)}{(L+y)^2}+(\kappa+\nu^\flat)O\left(L^{-5/2}\right).$$

where $\chi(\mu)$ is given by

$$\chi(\mu)=\left[\mu_2-\frac{9\mu_3}{8}\right]\left[\frac{\pi^2\mu_2}{4}-\frac{3\pi^2\mu_3}{16}+\frac{9\mu_3}{16}\right]\frac{1}{\left(2\mu_2-15\mu_3/8\right)^2}. \tag{6.5}$$

Thus, the skew stabilization takes place if

$$\chi(\mu)>0$$

The critical value ν_c for which the stabilization takes place is given by

$$\nu_c\sim\nu\frac{2\chi(\mu)}{3L^2}. \tag{6.6}$$

We find the value given at proposition 5.8, when $\mu_2=1$ and the other μ_j's vanish.

When the coefficient m of $-\partial^2/\partial x^2$ is replaced by $m+i\sqrt{\varepsilon}\mu_0$, the skew stabilization also takes place under the same condition and for the same critical value of ε, up to higher order terms.

Therefore, we have proved the final result of this article:

Proposition 6.1. *In the general case i.e. for the equation*

$$u_t=\left(m+i\sqrt{\varepsilon}\,\mu_0\right)u_{xx}-\left(m+i\sqrt{\varepsilon}\,\mu_1\right)u+\left(1+i\sqrt{\varepsilon},\mu_2\right)|u|^2u-\left(1+i\sqrt{\varepsilon}\mu_3\right)|u|^4u,$$

there exists a pulse-like solution for all small enough ν, for all $p>0$, and for all $y\le L^p$; provided that $\chi(\mu)$ defined by (6.5) is strictly positive; in this case, the skew stabilization takes place, at a critical parameter ν_c whose equivalent is given by (6.6).

7. Appendix. An implicit function Theorem with estimates

The purpose of this appendix is to prove a version of the implicit function Theorem which is appropriate for singular perturbations.

Lemma 7.1. *Let X and Z be Banach spaces, and let f be a C^2 function from a neighborhood $\mathcal{U}$ of $x_0 \in X$ to Z. Let $z_0 = f(x_0)$. Assume that $A = Df(x_0)$ has a bounded inverse A^{-1}. Assume that the ball of radius ρ and of center x_0 is included in $\mathcal{U}$. Let*

$$M = \sup_{|\xi| \le \rho} \|A^{-1} D^2 f(x_0 + \xi)\|.$$

There exist constants a and K given by

$$a = \min(1, (2\rho M)^{-1}), \quad K = \frac{3a\rho}{4}$$

such that if $|A^{-1} z_0| \le K$, the equation

$$f(x) = 0$$

possesses a unique solution in the ball $\{|x - x_0| \le a\rho\}$; moreover, this solution satisfies

$$|x - x_0| \le 2|A^{-1} z_0| \text{ and } |x - x_0 + A^{-1} z_0| \le 2M |A^{-1} z_0|^2.$$

Proof. Let

$$F(\xi, t) = \xi - A^{-1} f(x_0 + \xi) + (1 - t) A^{-1} z_0.$$

It is equivalent to find a fixed point of F for $t = 1$ and to find a solution of $f(x) = 0$. Introducing F enables us to start from $t = 0$, where we have $F(0, 0) = 0$. The implicit function Theorem in its classical formulation would give us the existence of an interval of t on which we can find a solution of $\xi - F(\xi, t) = 0$, but it would not ensure that we have a solution up to $t = 1$. The purpose of our estimates is to show that we can go as far as $t = 1$.

Let us determine $a \in (0, 1]$ such that $\xi \mapsto F(\xi, t)$ is a contraction of ratio $1/2$ for $|\xi| \le a\rho$ and $0 \le t \le 1$:

$$\begin{aligned}
&F(\xi', t) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -F(\xi, t) \\
&= \xi' - \xi - A^{-1} \int_0^1 Df\big(x_0 + \xi + t(\xi' - \xi)\big)(\xi' - \xi)\, dt \\
&= \xi' - \xi - A^{-1} \int_0^1 \Big[Df(x_0)(\xi' - \xi) \\
&\quad + \int_0^1 D^2 f\big(x_0 + s(\xi + t(\xi' - \xi))\big)(\xi + t(\xi' - \xi)) \otimes (\xi' - \xi)\, ds \Big]\, dt \\
&= -\int_0^1 \int_0^1 D^2 f\big(x_0 + s(\xi + t(\xi' - \xi))\big)(\xi + t(\xi' - \xi)) \otimes (\xi' - \xi)\, ds\, dt.
\end{aligned}$$

Therefore, we have the inequality

$$|F(\xi', t) - F(\xi, t)| \le M a\rho |\xi' - \xi|.$$

Thus, the first condition that we wish to impose is

$$M a \rho \le \frac{1}{2}.$$

The second step is to impose a condition on a and on $|B^{-1}z_0|$ such that $\xi \mapsto F(\xi,t)$ maps the ball of center x_0 and radius $a\rho$ into itself, for $t \leq 1$. We have

$$\begin{aligned} |F(\xi,t)| &\leq \left|\xi - A^{-1}\left[Df(x_0)\xi + \int_0^1 D^2 f(x_0+t\xi)\xi^{\otimes 2}(1-t)\,dt\right] - tA^{-1}z_0\right| \\ &\leq Ma^2\rho^2 2 + |A^{-1}z_0|. \end{aligned}$$

Therefore, it is enough to require that

$$|A^{-1}z_0| \leq \frac{3a\rho}{4}.$$

Thanks to the strict contraction Theorem, there exists for $t \in [0,1]$ and for $|x| \leq a\rho$ a unique solution of

$$F(\xi,t) = \xi.$$

Let us denote this solution by $\xi = g(t)$. We may estimate $g(t)$ as follows: we have the inequality

$$|g(t)| \leq |F(g(t),t) - F(0,t)| + |F(0,t)| \leq \frac{1}{2}|g(t)| + t|A^{-1}z_0|.$$

Therefore,

$$|g(t)| \leq 2t|A^{-1}z_0|.$$

But we can obtain a much better estimate, since

$$\begin{aligned} g(t) + tA^{-1}z_0 &= F(g(t),t) + tA^{-1}z_0 \\ &= g(t) - A^{-1}f(x_0+g(t)) + (1-t)A^{-1}z_0 + tA^{-1}z_0 \\ &= g(t) - A^{-1}z_0 - A^{-1}Df(x_0)g(t) \\ &\quad - \int_0^1 D^2 f(x_0+sg(t))g(t)^{\otimes 2}(1-s)\,ds + A^{-1}z_0 \\ &= -\int_0^1 D^2 f(x_0+sg(t))g(t)^{\otimes 2}(1-s)\,ds. \end{aligned}$$

Therefore,

$$|g(t) + tA^{-1}z_0| \leq \frac{M}{2}|g(t)|^2 \leq 2M|A^{-1}z_0|^2,$$

which concludes the proof. □

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